Bootstrapping out-of-sample predictability tests with real-time data*

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Abstract

In this paper we develop a block bootstrap approach to out-of-sample inference when real-time data are used to produce forecasts. In particular, we establish its first-order asymptotic validity for West-type (1996) tests of predictive ability in the presence of regular data revisions. This allows the user to conduct asymptotically valid inference without having to estimate the asymptotic variances derived in Clark and McCracken's (2009) extension of West (1996) when data are subject to revision. Monte Carlo experiments indicate that the bootstrap can provide satisfactory finite sample size and power even in modest sample sizes. We conclude with an application to inflation forecasting that adapts the results in Ang et al. (2007) to the presence of real-time data.

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1 Introduction

Real-time vintage data are often used to construct forecasts. At central banks, such as the Federal Reserve and the European Central Bank, this is due to the real-time nature of their forecasting problem. For example, at each FOMC briefing and General Council meeting, it is expected that the staff will have taken on board the latest data when constructing their forecasts. This is important not only because new data are continuously being released but also because previously released macroeconomic data are revised and the revised data are generally considered more accurate. Documenting the importance of real-time data to central bank forecasting was one of the main motivations for the development of the Real-Time Dataset for Macroeconomists (RTDSM) at the Federal Reserve Bank of Philadelphia (Croushore and Stark, 2001). In particular, while Stark and Croushore (2002) and Croushore (2006) emphasize that data revisions obviously change the conditioning information available to the forecasting agent, the revisions can affect parameter estimates and even the functional form of the predictive model.

With this in mind, there is an increasing emphasis on evaluating central bank-, survey-, and model-based forecasts using real-time vintage data. For example, Faust and Wright (2009) evaluate the efficiency of Greenbook forecasts using a sequence of historical datasets archived at the Board of Governors. Faust, Rogers, and Wright (2005) investigate prospects for real-time exchange rate forecasting using vintages of data available at the OECD's database of Main Economic Indicators. Chauvet and Piger (2008) illustrate the real-time accuracy of recession forecasts using the ALFRED database hosted by the Federal Reserve Bank of St. Louis. Croushore (2011) provides a review of other issues related to real-time forecasting and monetary policy evaluation in the context of the RTDSM. While much of this empirical literature focuses on the US or the Euro area, Garratt et al. (2011) investigate real-time forecasting issues associated with data for Australia, New Zealand, and Norway.

Unfortunately, while the empirical forecasting literature has adapted to the real-time nature of macroeconomic data, the bulk of the theoretical literature on forecast evaluation largely ignores it. Examples like West (1996), Clark and McCracken (2001), Corradi, Swanson, and Olivetti (2001), Giacomini and Rossi (2010), and Odendahl, Rossi and Sekhposyan (2022) each ignore the possibility that at any given forecast origin the most recent data may be subject to revision. This is an issue because out-of-sample tests of predictive ability are operationally distinct from in-sample tests, in ways that make out-of-sample tests particularly susceptible to changes in the correlation structure of the data as the revision process unfolds. This susceptibility has three sources: (i) while parameter estimates are typically functions of only a small number of observations that remain subject to revision, out-of-sample statistics are functions of a sequence of parameter estimates (one for each forecast origin

$t = R, \ldots, T$), (ii) the predictand used to generate the forecast and (iii) the dependent variable used to construct the forecast error may be subject to revision, and hence a sequence of revisions contribute to the test statistic. If data subject to revision possess a different mean and covariance structure than final revised data (e.g., Aruoba (2008)), tests of predictive ability using real-time data may have a different asymptotic distribution than tests constructed using data that are never revised.

In the context of OLS-estimated linear models, Clark and McCracken (2009) show that real-time data can affect tests of predictive ability. In particular, they rederive the results in West (1996) but allow for forecasts that are constructed sequentially across vintages of real-time data. They find that data revisions can lead to substantial changes in asymptotic variances for asymptotically normal tests. In addition, they find that in the context of nested model comparisons, tests can be asymptotically normal even when they were not in the absence of revisions. Subsequently, the analytical results indicate that ignoring the real-time nature of the data can lead to large size distortions and substantial reductions in power. These distortions can arise when revisions are best categorized as news, in the sense of Mankiw, Runkle, and Shapiro (1984), but are most likely to occur when the revisions contain at least some element of noise.

The main contribution of this paper is to provide a bootstrap approach to conducting inference in the same framework as Clark and McCracken (2009). When appropriately implemented, the bootstrap allows for valid out-of-sample inference without having to estimate the somewhat complicated asymptotic variances derived by Clark and McCracken (2009). More specifically, we provide analytical, Monte Carlo and empirical evidence on the effectiveness of a new block bootstrap approach to out-of-sample inference when forecasts are constructed using real-time vintage data. In many ways our bootstrap is unique. There exists no other bootstrap specifically designed for vintage data. Our block bootstrap treats those observations near the end of a given vintage of data as fundamentally distinct from older, fully revised observations. By doing so, we are able to mimic the triangular array of real-time data in each bootstrap sample.

While our bootstrap is new, it clearly builds on previous work, including Calhoun (2015) who develops a bootstrap that can be used for conducting inference on asymptotically normal out-of-sample tests of predictive ability. The procedure allows for estimation error to contribute to the asymptotic distribution as derived in West (1996) but only in the same environment as West - one that does not allow for real-time vintages of data. It is also related to that of Corradi and Swanson (2007) who propose a block bootstrap procedure for predictive inference based on recursive estimation schemes. Like they do, we must account for the fact that as we move across forecast origins the sample size increases and hence some observations are used more frequently than others, and this affects the design of the bootstrap statistic (in particular, it requires a careful choice of the centering constant).

Monte Carlo simulations indicate that our bootstrap can provide accurately sized tests of predictive ability in the presence of revisions. Even so, the bootstrap algorithm has limitations and the assumptions underlying its validity are obvious approximations to reality. Specifically, we assume that revisions are finite lived. For many macroeconomic variables like employment, consumption, and industrial production, as we move from one vintage to the next only a small handful of the most recent observations are revised - and align with our assumptions. Where we differ is that we abstract from benchmark revisions. While this reduces the realism of the assumptions, it allows us to take one step toward accounting for data revisions in tests of predictive ability.

We then apply our bootstrap in the context of comparing the forecast accuracy of models used to forecast CPI- and PCE-based inflation. Specifically, we revisit a small subset of the results in Ang et al. (2007) in which they conclude that survey-based forecasts are more accurate than a variety of model-based forecasts. Our goal is to see if their conclusions are robust to the presence of real-time data, which they do not consider. Our results largely align with their conclusion, with the exception that the models seem to out-perform the surveys when forecasting PCE-based inflation at longer horizons. Even so it's worth noting that due to data availability, our sample is distinct from theirs.

The remainder of the paper proceeds as follows. Section 2 introduces the notation and describes the forecasting environment. Section 3 presents the assumptions. Section 4 delineates the asymptotic distribution of the proposed test statistic. Section 5 describes the bootstrap algorithms and characterizes the asymptotic properties of our bootstrap approach to inference. Section 6 presents Monte Carlo evidence on the finite sample size and power of the test using both asymptotic critical values as well as bootstrap critical values. Section 7 illustrates our bootstrap approach to inference in the context of real-time inflation forecasting. Section 8 concludes. An Appendix contains proofs of the theoretical results.

2 Framework

We follow the revision structure described in Clark and McCracken (2009). At each forecast origin $t = R, \ldots, T$, forecasts of a scalar target variable y are made using a finite dimensioned vector of predictors x based on the current vintage of data $\{y_s(t), x_s(t) : s = 1, \ldots, t\}$. We assume that after a finite number of r releases the revisions are final. For example, we let $y_{s|j}$ denote the j^{th} release of y_s, and hence $y_{s|1}$, $\ldots$, $y_{s|r-1}$ are all preliminary values of y_s while $y_{s|r}$ corresponds to the final value

Table 1: Structure of real-time data with $r = 2$

	Vintage date (t)			
Obs. s	R	$R+1$	$\dots$	$T+1$
1	y_1	y_1		y_1
2	y_2	y_2		y_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$R-2$	y_{R-2}			
$R-1$	y_{R-1}	y_{R-1}		y_{R-1}
R	$y_{R\|1}$	y_R		y_R
$R+1$		$y_{R+1\|1}$		$\vdots$
$\vdots$				
T				y_T
$T+1$				$y_{T+1\|1}$

of y_s, which we write as $y_{s|r} = y_s$. For each t,

$$y_s(t) = \begin{cases} y_s & \text{for } 1 \leq s \leq t - r + 1 \\ y_{s|j} & \text{for } s = t - j + 1, \quad j = r-1, \dots, 1. \end{cases}$$

To illustrate the notation, consider the case of a single revision with $r = 2$. Table 1 provides a description of this data structure. It shows that the data has a triangular structure, where the last observation in each column (vintage) is updated in the following vintage.

The τ-step ahead forecast is formed using linear OLS estimated models $x_t'(t)\hat{\beta}(t)$ where

$$\hat{\beta}(t) \equiv \left(\sum_{s=1+\tau}^{t} x_{s-\tau}(t)\, x_{s-\tau}'(t) \right)^{-1} \sum_{s=1+\tau}^{t} x_{s-\tau}(t)\, y_s(t),$$

and the bracket notation in $\hat{\beta}(t)$ is used to emphasize dependence on vintage t data. The forecasts are evaluated against $y_{t+\tau|r'}$, the r'th release of the target variable $y_{t+\tau}$, where $r' \in \{1, \dots, r\}$. For instance, if $\tau = 1$ and $r' = 1$, the forecast is evaluated against the first release of y_{t+1}, $y_{t+1|1} = y_{t+1}(t+1)$. But if $r' = 2$, the forecast is evaluated against $y_{t+1|2} = y_{t+1}(t+2)$, the second release of y_{t+1} where $y_{t+1}(t+2) = y_{t+1}$ if $r = 2$.

Given a sequence of real-time forecasts, one is interested in testing the scalar null hypothesis

$$H_0 : Ef\left(y_{t+\tau|r'}, x_t(t), \beta_0\right) = 0,$$

for a known function $f(.)$, which we will often simplify as $f_{t+\tau|r'} = f\left(y_{t+\tau|r'}, x_t(t), \beta_0\right)$. Looking ahead toward our bootstrap, note that $f_{t+\tau|r'}$ depends on data from two different vintages: the vintage containing $y_{t+\tau|r'}$ and the vintage containing $x_t(t)$.

As a simple example, for a test of zero-mean prediction error $Eu_{t+\tau|r'} = 0$, we have $f_{t+\tau|r'} = u_{t+\tau|r'} = y_{t+\tau|r'} - x_t'(t)\,\beta_0$. A more complex example is a test of equal predictive ability between two models. For example, let $x'_{i,t}(t)\hat{\beta}_i(t)$, $i = 1, 2$, denote forecasts that we evaluate under quadratic loss. In this case, the null is $E(u^2_{1,t+\tau|r'} - u^2_{2,t+\tau|r'}) = 0$ and the subsequent function is

$$f_{t+\tau|r'} = \left(y_{t+\tau|r'} - x'_{1,t}(t)\,\beta_{1,0}\right)^2 - \left(y_{t+\tau|r'} - x'_{2,t}(t)\,\beta_{2,0}\right)^2,$$

where we have now defined $\beta_0 = (\beta'_{1,0}, \beta'_{2,0})'$ and $x_t(t) = (x'_{1,t}(t), x'_{2,t}(t))'$.

To test the null hypothesis we form a test statistic based on the finite sample analogue of $f_{t+\tau|r'}$, where β_0 is replaced by $\hat{\beta}(t)$:

$$\hat{S}_P = P^{-1/2} \sum_{t=R}^{T} f\left(y_{t+\tau|r'}, x_t(t), \hat{\beta}(t)\right),$$

where $P = T - R + 1$. This notation implies that we have a total of $P + \tau + r' - 1$ vintages. The first P vintages are used to produce τ-step-ahead forecasts while $P - \tau - r'$ of these vintages are also used to evaluate the forecasts. The final $\tau + r' - 1$ vintages are used only for forecast evaluation.

In the context of tests of equal predictive ability between two OLS estimated linear models, Clark and McCracken (2009) show that $\hat{S}_P$ can be asymptotically normal with an asymptotic variance reminiscent of that developed in West (1996). Their method of proof, which we will follow, takes advantage of the assumption that the revision process is finite lived. This is a useful approximation because it implies that under additional mixing and moment conditions, $\hat{\beta}(t)$ is asymptotically equivalent to the estimator that uses fully revised data only,

$$\hat{\beta}_t \equiv \left(\sum_{s=1+\tau}^{t} x_{s-\tau}x'_{s-\tau}\right)^{-1} \sum_{s=1+\tau}^{t} x_{s-\tau}y_s.$$

Absent this assumption, we would have to consider the possibility that the probability limit of $\hat{\beta}(t)$, $\beta_0 = (E(x_{s-\tau}x'_{s-\tau}))^{-1}E(x_{s-\tau}y_s)$, would vary across forecast origins t due to the revision process itself rather than some underlying changes in the data-generating process for the fully revised data. With this in mind, in the following two sections we first extend the results in Clark and McCracken (2009) to a wider range of functions $f_{t+\tau|r'}$. We then delineate our bootstrap algorithm and show how it allows us to replicate the null asymptotic distribution of $\hat{S}_P$ and thus be able to conduct inference using a percentile bootstrap and avoid the need to estimate the somewhat complicated asymptotic variances.

3 Assumptions

In this section, we introduce the assumptions that allow us to obtain the asymptotic distribution of $\hat{S}_P$. Throughout we let $f_{t+\tau|r',\beta} \equiv \frac{\partial}{\partial \beta'} f\left(y_{t+\tau|r'}, x_t(t), \beta_0\right)$ and define $F \equiv Ef_{t+\tau|r',\beta}$. Our assumptions are comparable to those in Clark and McCracken (2009), which adapt the results in West (1996) to environments where revisions are present.

Assumption 1 *In an open neighborhood $\mathcal{N}$ around β_0 and with probability 1, (a) $f_{t+\tau|r'}(\beta)$ is measurable and twice continuously differentiable. (b) There exists a constant $D<\infty$ such that for all t, $\sup_{\beta\in\mathcal{N}}\left|\frac{\partial^2 f_{t+\tau|r'}(\beta)}{\partial\beta\partial\beta'}\right|<m_{t+\tau}$ with a measurable function $m_{t+\tau}$ such that $E(m_{t+\tau})<D$.*

Assumption 1 ensures that the function $f_{t+\tau|r'}$ is well approximated by a quadratic function in the neighborhood of β_0. Without data revisions, it corresponds to Assumption 1 in West (1996). As West (1996) remarks, this assumption is automatically satisfied if the function is a squared forecast error and the model is linear, provided second moments of the target variable $y_{t+\tau|r'}$ and predictors $x_t(t)$ exist.

Assumption 2 *For each model $i=1,\ldots,k$, where k is finite, the following holds. (a) The final-data estimate $\hat{\beta}_{i,t}$ satisfies $\hat{\beta}_{i,t}-\beta_{i,0}=B_i(t)H_i(t)$, where*

$$B_i(t)=\left(t^{-1}\sum_{s=1+\tau}^{t}x_{i,s-\tau}x'_{i,s-\tau}\right)^{-1}\xrightarrow{a.s.}B_i,\ \text{and } H_i(t)=t^{-1}\sum_{s=1+\tau}^{t}h_{i,s}\ \text{with } E(h_{i,s})=0,$$

where $B_i=\left(E(x_{i,s}x'_{i,s})\right)^{-1}$ and $h_{i,s}=x_{i,s-\tau}(y_s-x'_{i,s-\tau}\beta_{i,0})$. (b) The real-time data estimate $\hat{\beta}_i(t)$ satisfies $\hat{\beta}_i(t)-\beta_{i,0}=\hat{B}_i(t)\hat{H}_i(t)$, where

$$\hat{B}_i(t)=\left(t^{-1}\sum_{s=1+\tau}^{t}x_{i,s-\tau}(t)x_{i,s-\tau}(t)'\right)^{-1}\ \text{and } \hat{H}_i(t)=t^{-1}\sum_{s=1+\tau}^{t}h_{i,s}(t),$$

with $h_{i,s}(t)=x_{i,s-\tau}(t)(y_s(t)-x_{i,s-\tau}(t)'\beta_{i,0})$.

The first part of Assumption 2 is a special case of Assumption 2 in West (1996) when the model parameters are estimated by OLS using fully revised data. The second part of Assumption 2 defines the real-time estimator $\hat{\beta}(t)$ as an OLS estimator based on vintage t data. Clark and McCracken (2009) rely on a similar assumption (see in particular their Assumption A1). Note that we focus exclusively on the recursive scheme when estimating model parameters. As we discuss later in the context of the bootstrap algorithm, the choice of centering constant depends explicitly on this restriction.

While our results allow for multiple models, as would be the case in a test of equal forecast accuracy, it is convenient to consolidate notation to a single parameter vector $\beta_0=(\beta'_{1,0},\ldots,\beta'_{k,0})'$ and let $x_t=(x'_{1,t},\ldots,x'_{k,t})'$. Having done so we define $\hat{\beta}_t$ so that $\hat{\beta}_t-\beta_0=B(t)H(t)$ where $B(t)=diag(B_i(t))$, $B=diag(B_i)$, $H(t)=(H'_1(t),\ldots,H'_k(t))'$, and $h_s=(h'_{1,s},\ldots,h'_{k,s})'$. The corresponding notation for $\hat{\beta}(t)$ is analogous so that $\hat{\beta}(t)-\beta_0=\hat{B}(t)\hat{H}(t)$.

Our next assumption is a moment and dependence assumption on the vector

$$g_{t+\tau|r'} \equiv \begin{pmatrix} \left(f_{t+\tau|r'} - Ef_{t+\tau|r'}\right) & \left(f_{t+\tau|r',\beta} - F\right) & h'_{t+\tau} & x'_t - Ex'_t \end{pmatrix}'.$$

Assumption 3 *(a) For some $d > 1$ and $\delta > 0$, $\sup_t E\left\|g_{t+\tau|r'}\right\|^{4d+\delta} < \infty$. (b) $g_{t+\tau|r'}$ is covariance stationary. (c) $\{g_{t+\tau|r'}\}$ is strong mixing with mixing coefficients of size $\frac{-3d}{d-1}$. (d) Ω is positive, where $\Omega \equiv \lim_{P,R\to\infty} Var(P^{-1/2}\sum_{t=R}^{T}\left(f_{t+\tau|r'} - Ef_{t+\tau|r'}\right) + FBP^{-1/2}\sum_{t=R}^{T} H(t))$.*

Assumption 3 modifies Assumption 3 in West (1996) to the context of real-time data. In addition, we strengthen the moment bound by an additional $\delta > 0$, which we use when proving our bootstrap results. This type of strengthening is common in the bootstrap literature, see e.g., Fitzenberger (Theorem 3.1, 1998). Similarly to Clark and McCracken (2009) (see their Assumption A2), we impose a mixing-type condition on the vector $g_{t+\tau|r'}$ which depends on the vintage horizon r' through the function $f_{t+\tau|r'}$ and its derivatives. We also require that Ω is positive as in Assumption A2 (f) of Clark and McCracken (2009). For many applications, like pairwise tests of equal accuracy between non-nested models, this restriction seems unnecessary. However, in the context of nested model comparisons, it is straightforward to show that whether Ω is positive depends on the properties of the revision process – a point we return to later.

Assumption 4 *For some $d > 1$, $r < \infty$ and $j = 1, \ldots, r$, $(y_{t|j}, x'_{t|j})'$ is uniformly L^{4d}-bounded.*

Assumption 4 is the same as Assumption A2(d) of Clark and McCracken (2009). It puts restrictions on the revision process. More specifically, it restricts the total number of releases to be a finite number r. It also requires the released data to have finite moments of order slightly larger than 4.[1] This implicitly restricts the stochastic order of magnitude of the revisions. As noted in the previous section, we rely on this assumption since it implies that $\hat{\beta}_t$ and $\hat{\beta}(t)$ are asymptotically equivalent.

Assumption 5 *$R, P \to \infty$ and $\lim_{P,R\to\infty} \frac{P}{R} \equiv \pi$, where $0 \leq \pi < \infty$.*

Assumption 5 is the same as Assumptions A4 and A4′ of Clark and McCracken (2009).

4 Asymptotic results

In this section we establish the asymptotic distribution of $\hat{S}_P$. To do so, we first show that $\hat{S}_P$ is asymptotically equivalent to

$$\tilde{S}_P = P^{-1/2}\sum_{t=R}^{T} f\left(y_{t+\tau|r'}, x_t(t), \hat{\beta}_t\right),$$

[1]Here and in what follows, we say that X_t is L^p-bounded if $(E|X_t|^p)^{1/p} < \infty$.

a test statistic based on $\hat{\beta}_t$ rather than $\hat{\beta}(t)$. Specifically, we prove the following result.

Lemma 4.1 *Under Assumptions 1-5,* $\hat{S}_P = \tilde{S}_P + o_p(1)$.

By eq. (4.1) of West (1996), adapted to the presence of vintage data, we can expand $\tilde{S}_P - P^{1/2}E(f_{t+\tau|r'})$ as

$$\tilde{S}_P - P^{1/2}E(f_{t+\tau|r'}) = P^{-1/2}\sum_{t=R}^{T}(f_{t+\tau|r'} - E(f_{t+\tau|r'})) + FBP^{-1/2}\sum_{t=R}^{T}H(t) + o_p(1) \equiv S_{1P} + FBS_{2P} + o_p(1).$$

The first term S_{1P} is equal to the scaled average of the demeaned real-time function $f_{t+\tau|r'}$ evaluated at β_0. This piece is asymptotically normally distributed by a central limit theorem (CLT) for strong mixing data, given our assumptions. The second term depends on S_{2P}, the scaled average of $H(t)$, the average of the scores h_s used in estimating β_0 based on the fully revised data. This piece captures the contribution of the parameter estimation uncertainty and can also be shown to be asymptotically normal, jointly with S_{1P}.

More specifically, following West (1996), S_{1P} and S_{2P} are jointly asymptotically normal, implying that

$$\tilde{S}_P - P^{1/2}E(f_{t+\tau|r'}) \xrightarrow{d} N(0, \Omega),$$

where

$$\Omega = \Omega_1 + FB\Omega_2 B'F' + \Omega_{12}B'F' + FB\Omega_{12}', \tag{1}$$

with

$$\Omega_1 \equiv \lim_{R,P\to\infty} Var(S_{1P}), \ \Omega_2 \equiv \lim_{R,P\to\infty} Var(S_{2P}), \ \text{ and } \ \Omega_{12} \equiv \lim_{R,P\to\infty} Cov(S_{1P}, S_{2P}).$$

The form of Ω is notationally the same as the one obtained in West (1996) without data revisions. One main difference is that Ω_1 is now the long-run variance of the scaled average of the real-time function $f_{t+\tau|r'}$ rather than the long-run variance of $f_{t+\tau} \equiv f(y_{t+\tau}, x_t, \beta_0)$, the function associated with fully revised data. Under Assumption 4, the parameter estimation uncertainty as measured by Ω_2 is the same as when all data used in estimation are final. However, its contribution to the overall covariance matrix Ω is different with data revisions. This is because $F \equiv E\left(f_{t+\tau|r',\beta}\right)$ is not necessarily equal to $E(f_{t+\tau,\beta}) \equiv E(\partial f(y_{t+\tau}, x_t, \beta_0)/\partial\beta')$.

The form of Ω shows that the covariance between S_{1P} and S_{2P} need not be asymptotically zero. While perhaps not immediately obvious, this has ramifications for how we design the bootstrap. In particular we take advantage of the fact that S_{2P} can be decomposed into the sum of two asymptotically uncorrelated terms, one of which is not correlated with S_{1P}. Borrowing from page 1081 of West (1996), we write S_{2P} as

$$S_{2P} \equiv P^{-1/2}\sum_{t=R}^{T} H(t) = P^{-1/2}\sum_{s=1+\tau}^{R} a_{R,0}h_s + P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}h_{R+i} \equiv S_{2P.1} + S_{2P.2},$$

where the weights $a_{R,i}$ are defined as $a_{R,i} \equiv \frac{1}{R+i} + \ldots + \frac{1}{R+P-1}$ for $0 \leq i \leq P-1$.

Lemma 4.2 *Under Assumptions 1-5,*

(a) $\Omega_2 \equiv \lim_{R,P\to\infty} Var\left(S_{2P}\right) = \Omega_{2.1} + \Omega_{2.2}$, *where*

$$\begin{aligned}
\Omega_{2.1} &\equiv \lim_{R,P\to\infty} Var\left(S_{2P.1}\right) = \lim_{R,P\to\infty} Var(P^{-1/2}\sum_{s=1}^{R} a_{R,0}h_s), \\
\Omega_{2.2} &\equiv \lim_{R,P\to\infty} Var\left(S_{2P.2}\right) = \lim_{R,P\to\infty} Var(P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}h_{R+i}).
\end{aligned}$$

(b) $\Omega_{12} \equiv \lim_{R,P\to\infty} Cov\left(S_{1P}, S_{2P}\right)$ *is equal to*

$$\Omega_{12} = \lim_{R,P\to\infty} Cov\left(S_{1P}, S_{2P.2}\right) = \lim_{R,P\to\infty} Cov\left(P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r'}, P^{-1/2}\sum_{s=1}^{P-1} a_{R,s}h_{R+s}\right).$$

Lemma 4.2 (a) shows that $S_{2P.1}$ is asymptotically uncorrelated with $S_{2P.2}$. Thus, the asymptotic variance of S_{2P} is the sum of the asymptotic variances of $S_{2P.1}$ and $S_{2P.2}$. Lemma 4.2 (b) shows that the asymptotic covariance between S_{1P} and $S_{2P.1}$ is zero. This implies that the covariance between S_{1P} and S_{2P} depends only on the covariance between S_{1P} and $S_{2P.2}$. As noted above, this proves useful when developing our bootstrap method in the next section.

5 Bootstrap results

Here we present a novel bootstrap algorithm and prove its first-order asymptotic validity when used for out-of-sample inference with real-time data. To develop intuition, we first describe our bootstrap algorithm for out-of-sample evaluation of one-step-ahead forecasts based on a simple location model where the data are subject to one single revision. We then extend these results to forecast evaluation based on general linear models with forecast horizons possibly larger than one and multiple (but finite) revisions.

5.1 A simple location model

Consider first the following location model:

$$y_t = \beta_0 + u_t,\ t = 1, 2, \ldots,$$

where y_t is the fully revised observation. With $r = 2$ releases and one revision, we have

$$y_s(t) = \begin{cases} y_s & 1 \le s \le t-1 \\ y_{t|1} & s = t, \end{cases}$$

where $y_{t|1}$ denotes the preliminary (or first release) of the value of y_t according to vintage t. This data structure is the one described in Table 1.

At each forecast origin $t = R, R+1, \ldots, T$, we forecast $y_{t+1}(t+1) = y_{t+1|1}$, next period's value of the first release of y_{t+1}. Hence, $\tau = r' = 1$. The point forecast is $\hat{\beta}(t) = t^{-1}\sum_{s=1}^{t} y_s(t)$ and the null hypothesis is

$$H_0 : E\left(f_{t+1|1}\right) = 0,$$

where $f_{t+1|1} \equiv f\left(y_{t+1|1}, \beta_0\right)$. In this example, $f_{t+1|1}$ depends only on the first-released observations $y_{t+1|1}$. The test statistics $\hat{S}_P$ and $\tilde{S}_P$ are

$$\hat{S}_P = P^{-1/2}\sum_{t=R}^{T} f\left(y_{t+1|1}, \hat{\beta}(t)\right), \text{ and } \tilde{S}_P = P^{-1/2}\sum_{t=R}^{T} f\left(y_{t+1|1}, \hat{\beta}_t\right),$$

where $\hat{\beta}_t = t^{-1}\sum_{s=1}^{t} y_s$ is the estimate of β_0 based on the fully revised data.

When specialized to this example, the asymptotic expansion of $\tilde{S}_P$ is

$$\tilde{S}_P - P^{1/2}E(f_{t+1|1}) = S_{1P} + FS_{2P} + o_p(1), \tag{2}$$

where $F \equiv E\left(f_{t+1|1,\beta}\right) = E(\frac{\partial}{\partial\beta} f\left(y_{t+1|1}, \beta_0\right))$,

$$\begin{aligned} S_{1P} &= P^{-1/2}\sum_{t=R}^{T}(f_{t+1|1} - Ef_{t+1|1}), \text{ and} \\ S_{2P} &= P^{-1/2}\sum_{t=R}^{T} H(t) = P^{-1/2}\sum_{s=1}^{R} a_{R,0}h_s + P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}h_{R+i} \equiv S_{2P.1} + S_{2P.2}, \end{aligned}$$

where $h_s = y_s - \beta_0$ and the weights $a_{R,i}$ are as defined previously. In this simple example, S_{1P} depends on $\left\{y_{t+1|1} : t = R, \ldots, T\right\}$ whereas S_{2P} depends on $\{h_s = y_s - \beta_0 : s = 1, \ldots, T\}$.

Let $\tilde{S}_P^*$ denote a bootstrap version of the original test statistic (we provide more details below). The decision rule is to reject H_0 at level α if $|\tilde{S}_P| \ge c_{1-\alpha}^*$, where $c_{1-\alpha}^*$ is the $100(1-\alpha)^{th}$ percentile of the bootstrap distribution of $|\tilde{S}_P^*|$. To be valid, $\tilde{S}_P^*$ needs to replicate the asymptotic expansion of $\tilde{S}_P$. In particular, the bootstrap statistic needs to replicate the (zero) asymptotic mean and the asymptotic variance Ω of $\tilde{S}_P$.

Next, we propose a bootstrap algorithm that accomplishes this goal. Our algorithm relies on an application of the moving blocks bootstrap (MBB) of Künsch (1989) and Liu and Singh (1992), adapted to the out-of-sample forecasting context under data revisions. We write $\gamma_s \sim$ MBB from

$\{1, \ldots, R\}$ to denote a random index that is generated from the set $\{1, \ldots, R\}$ using the MBB. Similarly, we write $\eta_s \sim$ MBB from $\{R+1, \ldots, T+1\}$ to indicate that η_s is obtained by the MBB on the set $\{R+1, \ldots, T+1\}$. We describe below precisely how to generate these indices using the MBB.

Bootstrap algorithm for a location model

1. For $s = 1, \ldots, R$, let $\gamma_s \sim$ MBB from $\{1, \ldots, R\}$. For $s = R+1, \ldots, T, T+1$ generate $\eta_s \sim$ MBB from $\{R+1, \ldots, T+1\}$, independently of $\{\gamma_s\}$.

2. For each $t = R, R+1, \ldots, T$, compute

$$\hat{\beta}_t^* = t^{-1} \sum_{s=1}^{t} y_s^*,$$

where

$$y_s^* = \begin{cases} y_{\gamma_s} & \text{if } \ s = 1, \ldots, R, \\ y_{\eta_s} & \text{if } \ s = R+1, \ldots, T+1. \end{cases}$$

3. For each $t = R, R+1, \ldots, T$, let

$$y_{t+1|1}^* \equiv y_{t+1}^* \left(t+1\right) = y_{\eta_{t+1}|1}.$$

and set

$$f_{t+1|1}^* \left(\hat{\beta}_t^*\right) \equiv f\left(y_{t+1|1}^*, \hat{\beta}_t^*\right) = f\left(y_{\eta_{t+1}|1}, \hat{\beta}_t^*\right).$$

4. Compute

$$\tilde{S}_P^* \equiv P^{-1/2} \sum_{t=R}^{T} \left(f_{t+1|1}^* \left(\hat{\beta}_t^*\right) - f_{t+1|1}\left(\bar{\beta}_t\right)\right),$$

where

$$\bar{\beta}_t \equiv E^* \left(\hat{\beta}_t^*\right) = \frac{R}{t} \hat{\beta}_R + \frac{t-R}{t} \hat{\beta}_P,$$

with $\hat{\beta}_R \equiv R^{-1} \sum_{t=1}^{R} y_t$ and $\hat{\beta}_P \equiv P^{-1} \sum_{t=R}^{T} y_{t+1}$.

Step 1 is used to obtain the bootstrap analogs of the vintages. More specifically, we generate two sets of random indices: $\{\gamma_s : s = 1, \ldots, R\}$ is used to build the first R bootstrap observations $\{y_s^* = y_{\gamma_s} : s = 1, \ldots, R\}$ and $\{\eta_s : s = R+1, \ldots, T+1\}$ is used to build the remaining $T-R+1 = P$ observations. Since the data are assumed weakly dependent, we rely on the MBB to generate $\{\gamma_s\}$ and $\{\eta_s\}$. (These two sets are generated independently. We will explain below why this is not a problem for the validity of the bootstrap statistic.) More specifically, for a block size equal to l, and assuming that[2] $R = k_1 l$, we generate

$$I_1, \ldots, I_{k_1} \sim \text{i.i.d. Uniform on } \{1, \ldots, R - l + 1\}.$$

These k_1 random variables indicate the beginning of each block. Then, for each $i = 1, \ldots, k_1$ and $j = 1, \ldots, l$, we set

$$\gamma_{1+(i-1)l+(j-1)} = I_i + (j - 1),$$

from which we obtain

$$\{\gamma_s : s = 1, \ldots, R\} = \{\gamma_{1+(i-1)l+(j-1)} : i = 1, \ldots, k_1;\ j = 1, \ldots, l\}.$$

Similarly, we generate η_s from the set $\{R + 1, \ldots, T + 1\}$ using a MBB based on the same[3] block size l. These random indices are used to obtain the remaining P observations. In particular, letting $P \equiv T - R + 1 = k_2 l$, we generate k_2 uniform draws:

$$J_1, \ldots, J_{k_2} \sim \text{i.i.d Uniform on } \{R + 1, \ldots, T + 1 - l + 1\}.$$

For each $i = 1, \ldots, k_2$ and $j = 1, \ldots, l$, we set

$$\eta_{R+1+(i-1)l+(j-1)} = J_i + (j - 1),$$

from which we get $\{\eta_s : s = R + 1, \ldots, T + 1\} = \{\eta_{R+1+(i-1)l+(j-1)} : i = 1, \ldots, k_2;\ j = 1, \ldots, l\}$.

Table 2 provides a description of the bootstrap data structure. This table is the bootstrap analog of Table 1. In this table, for each vintage column t in $R + 1, \ldots, T + 1$, we set

$$y_s^*(t) = \begin{cases} y_{\gamma_s} & 1 \le s \le R \\ y_{\eta_s} & R + 1 \le s < t \\ y_{\eta_s|1} & s = t. \end{cases}$$

We see that except for the first column (vintage R), which sets all the observations in the bootstrap vintage R as final, all the remaining (vintages) columns replicate the triangular structure of the data.[4]

Note that the bootstrap observations indexed by η_s for $s = R + 1, \ldots, T + 1$ are used both in estimating β_0 (for vintages R+1 and beyond) as well as in evaluating the forecast. For this reason, these

[2] When R is not divisible by l, we let $k_1 = [R/l]$, the smallest integer that is greater or equal to R/l. We then obtain $R^* = k_1 l$ bootstrap observations. When $R^* \ge R$, we discard the observations in the last block so as to make the number of bootstrap observations equal to R.

[3] Assuming the same block size is for simplicity only. We could allow for different block sizes.

[4] The fact that we do not exactly replicate vintage R's structure is not a problem because this vintage is only used for estimation of β_0 at $t = R$. Hence, this is equivalent to estimating β_0 in this vintage using only final values, i.e. we obtain $\hat{\beta}_R^* = R^{-1} \sum_{s=1}^R y_s^*$ rather than $\hat{\beta}^*(R) = R^{-1} \sum_{s=1}^R y_s^*(R)$. Since these two estimates are asymptotically equivalent under the assumption of finite revisions, our approach remains valid.

Table 2: Structure of pseudo real-time data with $r = 2$

	Vintage date (t)			
Obs. s	R	$R+1$	$\cdots$	$T+1$
1	$y_1^* = y_{\gamma_1}$	$y_1^* = y_{\gamma_1}$		$y_1^* = y_{\gamma_1}$
2	$y_2^* = y_{\gamma_2}$	$y_2^* = y_{\gamma_2}$		$y_2^* = y_{\gamma_2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$R-1$	$y_{R-1}^* = y_{\gamma_{R-1}}$	$y_{R-1}^* = y_{\gamma_{R-1}}$		$y_{R-1}^* = y_{\gamma_{R-1}}$
R	$y_R^* = y_{\gamma_R}$	$y_R^* = y_{\gamma_R}$		$y_R^* = y_{\gamma_R}$
$R+1$		$y_{R+1\|1}^* = y_{\eta_{R+1}\|1}$		$y_{R+1}^* = y_{\eta_{R+1}}$
$\vdots$				
T				$y_T^* = y_{\eta_T}$
$T+1$				$y_{T+1\|1}^* = y_{\eta_{T+1}\|1}$

observations can be preliminary or final. This is the main reason we introduce a new random index η_s (generated independently of γ_s), which is randomly drawn from $\{R+1, \ldots, T+1\}$. Restricting the support of η_s to $\{R+1, \ldots, T+1\}$ ensures that both y_s and $y_{s|1}$ are available, implying that we can obtain their bootstrap analogs. If instead we had generated a single index η_s from the entire set $\{1, \ldots, R, R+1, \ldots, T+1\}$ (as in the bootstrap schemes of Corradi and Swanson (2007) and Calhoun (2015)), we would not be able to guarantee that a preliminary value is available for all observations. For instance, if $\eta_{R+1} = 1$, we do not observe $y_{1|1}$ and therefore cannot obtain $y_{R+1|1}^* = y_{\eta_{R+1}|1}$. The fact that we need to replicate the original vintage data structure is the crucial distinguishing feature of our paper and the main motivation for proposing a new bootstrap algorithm.

Step 2 obtains a bootstrap analogue of $\hat{\beta}_t$ using only final observations. Focusing on $\tilde{S}_P^*$ (which uses $\hat{\beta}_t^*$) rather than on the real-time bootstrap statistic $\hat{S}_P^*$ (which uses the real-time bootstrap estimate $\hat{\beta}^*(t)$) simplifies the application and the theory of the bootstrap. This approach is justified by Lemma 4.1, which shows the asymptotic equivalence of $\tilde{S}_P$ and $\hat{S}_P$.[5]

Step 3 creates the bootstrap observations used for forecast evaluation. Specifically, for $t = R, \ldots, T$, we let $y_{t+1|1}^* \equiv y_{t+1}^*(t+1) = y_{\eta_{t+1}|1}$ denote the bootstrap analog of $y_{t+1|1}$ and set

$$f_{t+1|1}^*\left(\hat{\beta}_t^*\right) \equiv f\left(y_{t+1|1}^*, \hat{\beta}_t^*\right) = f\left(y_{\eta_{t+1}|1}, \hat{\beta}_t^*\right).$$

This function is the bootstrap analog of $f_{t+1|1}\left(\hat{\beta}_t\right) \equiv f\left(y_{t+1|1}, \hat{\beta}_t\right)$.

Step 4 computes $\tilde{S}_P^*$, the bootstrap analog of $\tilde{S}_P$. This bootstrap test statistic centers $f_{t+1|1}^*\left(\hat{\beta}_t^*\right)$

[5]For vintages $t = R+1, \ldots, T+1$, an alternative estimator is $\hat{\beta}^*(t) = t^{-1}\sum_{s=1}^{t} y_s^*(t)$, the real-time bootstrap analog of $\hat{\beta}(t)$. Since the proofs of bootstrap validity are more involved for this estimator, we do not consider this alternative here.

around $f_{t+1|1}\left(\bar{\beta}_t\right)$, where[6]

$$\bar{\beta}_t \equiv E^*\left(\hat{\beta}_t^*\right) = \frac{R}{t}\hat{\beta}_R + \frac{t-R}{t}\hat{\beta}_P, \text{ for } t = R, R+1, \ldots, T,$$

with $\hat{\beta}_R \equiv R^{-1}\sum_{t=1}^{R} y_t$ and $\hat{\beta}_P \equiv P^{-1}\sum_{t=R}^{T} y_{t+1}$. Note that both $\hat{\beta}_R$ and $\hat{\beta}_P$ converge in probability to β_0 since $R, P \to \infty$ jointly. This implies that $\bar{\beta}_t \to_p \beta_0$ for $t = R, \ldots, T$.

A naive application of the MBB to the out-of-sample test statistic $\tilde{S}_P$ would suggest we compute

$$S_{P,naive}^* \equiv P^{-1/2}\sum_{t=R}^{T}\left(f_{t+1|1}^*\left(\hat{\beta}_t^*\right) - f_{t+1|1}\left(\hat{\beta}_t\right)\right).$$

However, this naive bootstrap statistic is not asymptotically valid. This was first remarked by Corradi and Swanson (2003, 2007) in a context without data revisions. The main reason is that recursive estimation of β_0 implies that earlier observations in the sample are used more frequently than subsequent observations. This implies that $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t^* - \hat{\beta}_t)$ does not mimic the distribution of $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t - \beta_0)$ when $\hat{\beta}_t$ is recursively estimated. Consequently, $\hat{\beta}_t$ no longer approximates $E^*(\hat{\beta}_t^*)$. Corradi and Swanson (2007) propose a bias correction method to recenter $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t^* - \hat{\beta}_t)$ appropriately. Our approach is different: we replace $\hat{\beta}_t$ by $\bar{\beta}_t$ when defining $\tilde{S}_P^*$, and we show that $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t^* - \bar{\beta}_t)$ mimics the limiting behavior of $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t - \beta_0)$ successfully. It's worth re-iterating that this result requires that the centering constant is designed for use under the recursive scheme. Under the rolling scheme, a distinct centering constant would be needed – an issue we do not pursue here.

Next, we explain heuristically why our bootstrap algorithm is asymptotically valid. Let $f_{t+1|1}^* \equiv f_{t+1|1}^*(\beta_0) = f\left(y_{\eta_{t+1}|1}, \beta_0\right)$. By considering two second-order mean value expansions of $f_{t+1|1}(\bar{\beta}_t)$ and $f_{t+1|1}^*(\hat{\beta}_t^*)$, both around β_0, we obtain the following stochastic expansion[7] of $\tilde{S}_P^*$:

$$\begin{aligned}
\tilde{S}_P^* &= P^{-1/2}\sum_{t=R}^{T}\left(f_{t+1|1}^*(\hat{\beta}_t^*) - f_{t+1|1}(\bar{\beta}_t)\right) \\
&= P^{-1/2}\sum_{t=R}^{T}\left(f_{t+1|1}^* - f_{t+1|1}\right) + FP^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t^* - \bar{\beta}_t) + o_p^*(1) \\
&\equiv S_{1P}^* + FS_{2P}^* + o_p^*(1),
\end{aligned}$$

where S_{1P}^* is the bootstrap analog of S_{1P}, and S_{2P}^* is the bootstrap analog of S_{2P}; this is the bootstrap analog of the asymptotic expansion of $\tilde{S}_P$. We can further decompose S_{2P}^* as follows. Since $\hat{\beta}_t^* = t^{-1}\sum_{s=1}^{t} y_s^*$ and $\bar{\beta}_t = t^{-1}\sum_{s=1}^{t} E^*(y_s^*)$, we have that

$$S_{2P}^* \equiv P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_t^* - \bar{\beta}_t) = P^{-1/2}\sum_{s=1}^{R} a_{R,0}h_s^* + P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}h_{R+i}^* \equiv S_{2P.1}^* + S_{2P.2}^*,$$

[6]Here and in the following, an asterisk appearing in E (and Var) denotes expectation (and variance) with respect to the bootstrap probability measure P^*, conditional on the original sample.

[7]See Appendix A for the definition of $o_p^*(1)$, as well as the definition of $\overset{d^*}{\to}$, which is used below.

where $h_s^* \equiv y_s^* - E^*\left(y_s^*\right)$.

We now show that $\tilde{S}_P^*$ is approximately centered at zero. First, since $E^*(h_s^*) = 0$ by construction, $E^*(S_{2P}^*) = 0$. Thus, the bootstrap distribution of $P^{-1/2} \sum_{t=R}^{T} (\hat{\beta}_t^* - \bar{\beta}_t)$ is centered at zero. This solves the bias problem discussed by Corradi and Swanson (2007) (who considered $\hat{\beta}_t$ rather than $\bar{\beta}_t$ when forming their bootstrap statistic in a context without data revisions). Second, we can show that $E^*\left(S_{1P}^*\right) = O_p\left(l/R\right) = o_p\left(1\right)$ if $l/R = o\left(1\right)$. This follows from standard MBB results (see e.g. Fitzenberger, 1998). Thus, $E^*(\tilde{S}_P^*)$ is asymptotically equal to zero.

We can also show that the bootstrap variance of $\tilde{S}_P^*$ mimics the asymptotic variance Ω. To see this, note that $S_{1P}^* \overset{d^*}{\to} N\left(0, \Omega_1\right)$. This follows from Theorem 3.1 of Fitzenberger (1998) given that $f_{t+1|1}^* = f\left(y_{t+1|1}^*, \beta_0\right) = f\left(y_{\eta_{t+1}|1}, \beta_0\right)$ is obtained by applying the MBB to the first released observations $\left\{y_{t+1|1} : t = R, \ldots, T\right\}$. Thus, our bootstrap mimics Ω_1.

Next, we explain why this bootstrap also mimics Ω_2. First, note that since we resample the first R observations independently of the last P observations, the covariance between $S_{2P.1}^*$ and $S_{2P.2}^*$ is zero in the bootstrap world. Thus, the bootstrap mimics the zero asymptotic covariance between $S_{2P.1}$ and $S_{2P.2}$ (which is established in Lemma 4.2(a)). Second, note that the fact that we use the MBB to obtain γ_s and η_s implies that the bootstrap variances of $S_{2P.1}^*$ and $S_{2P.2}^*$ converge to the long-run variances $\Omega_{2.1}$ and $\Omega_{2.2}$, respectively.

Finally, note that our bootstrap also captures the covariance between S_{1P} and $S_{2P.2}$. The independence between γ_s and η_s implies that the bootstrap analogs of S_{1P}^* and $S_{2P.1}^*$ are independent, but this is not a concern since the covariance between S_{1P} and $S_{2P.1}$ is asymptotically zero by Lemma 4.2(b).

5.2 Extension to linear models

In the general framework, we forecast $y_{t+\tau|r'}$ using a linear model with predictors $x_t\left(t\right)$, where the coefficients β_0 are estimated recursively by OLS. The location model is a special case of this setup where $x_s\left(t\right) \equiv 1$ for all s, t, with the difference that the target variable is $y_{t+\tau|r'}$ (rather than $y_{t+1|1}$). The forecast function is now $f_{t+\tau|r'} \equiv f\left(y_{t+\tau|r'}, x_t\left(t\right); \beta_0\right)$, where $x_t\left(t\right)$ contains the predictors for $y_{t+\tau|r'}$ available in vintage t. When lagged dependent variables are present, $x_t\left(t\right)$ may contain a mix of preliminary and final observations.

The main difference with respect to the simple location model is in the bootstrap estimation of β_0. For each vintage $t = R, \ldots, T$ we resample the "pairs" $z_s \equiv (y_s, x_{s-\tau}')'$ used in estimating β_0 at each forecast origin. As in the location model, we estimate $\hat{\beta}_t^*$ using only finally revised bootstrap data. While we state the algorithm in the context of a single model, the extension to multiple models is straightforward.

The bootstrap algorithm for forecasts based on general linear regression models is as follows.

Bootstrap algorithm for linear models

1. Let $R-(1+\tau)+1=k_1 l$ and generate $I_1,\dots,I_{k_1}\sim$ i.i.d. Uniform on $\{1+\tau,\dots,R-l+1\}$. Then, for each $i=1,\dots,k_1$ and $j=1,\dots,l$, set $I_i+(j-1)=\gamma_{1+\tau+(i-1)l+(j-1)}$ and let

$$\{\gamma_s : s=1+\tau,\dots,R\}=\{\gamma_{1+\tau+(i-1)l+(j-1)} : i=1,\dots,k_1;\; j=1,\dots,l\}.$$

Let $T+\tau-(R+1)+1=k_2 l$ and generate $J_1,\dots,J_{k_2}\sim$ i.i.d Uniform on $\{R+\tau,\dots,T+\tau-l+1\}$. For each $i=1,\dots,k_2$ and $j=1,\dots,l$, set $J_i+(j-1)=\eta_{R+1+(i-1)l+(j-1)}$, and let

$$\{\eta_s : s=R+1,\dots,T+\tau\}=\{\eta_{R+1+(i-1)l+(j-1)} : i=1,\dots,k_2;\; j=1,\dots,l\}.$$

2. For $t=R,\dots,T$, set

$$z_s^{*\prime}\equiv(y_s^*,x_{s-\tau}^{*\prime})=\begin{cases}(y_{\gamma_s},x'_{\gamma_s-\tau}) & 1+\tau\le s\le R\\ (y_{\eta_s},x'_{\eta_s-\tau}) & R+1\le s<t,\end{cases}$$

and compute

$$\hat\beta_t^*=\left(\frac{1}{t}\sum_{s=1+\tau}^{t}x_{s-\tau}^*x_{s-\tau}^{*\prime}\right)^{-1}\left(\frac{1}{t}\sum_{s=1+\tau}^{t}x_{s-\tau}^*y_s^*\right).$$

3. For $t=R,\dots,T$, let

$$(y^*_{t+\tau|r'},x_t^*(t)')=(y_{\eta_{t+\tau}|r'},x_{\eta_{t+\tau}-\tau}(\eta_{t+\tau}-\tau)'),$$

and compute

$$f^*_{t+\tau|r'}\left(\hat\beta_t^*\right)\equiv f\left(y^*_{t+\tau|r'},x_t^*(t)',\hat\beta_t^*\right).$$

4. Compute

$$\tilde S_P^*\equiv P^{-1/2}\sum_{t=R}^{T}\left(f^*_{t+\tau|r'}(\hat\beta_t^*)-f_{t+\tau|r'}(\bar\beta_t)\right),$$

where $\bar\beta_t=\frac{R}{t}\hat\beta_R+\frac{t-R}{t}\hat\beta_P$, with

$$\hat\beta_R=\left(\frac{1}{R}\sum_{s=1+\tau}^{R}x_{s-\tau}x'_{s-\tau}\right)^{-1}\left(\frac{1}{R}\sum_{s=1+\tau}^{R}x_{s-\tau}y_s\right)$$

and

$$\hat\beta_P=\left(\frac{1}{P}\sum_{s=R+\tau}^{T+\tau}x_{s-\tau}x'_{s-\tau}\right)^{-1}\left(\frac{1}{P}\sum_{s=R+\tau}^{T+\tau}x_{s-\tau}y_s\right).$$

Remark 1 *The presence of the predictors $x_t(t)$ when forecasting $y_{t+\tau|r'}$ creates some differences with respect to the simple location model's algorithm. The first difference is that we restrict the support of the MBB indices γ_s to $\{1+\tau,\ldots,R\}$ rather than $\{1,\ldots,R\}$. This is because the recursive estimates of β_0 depend on $(y_s, x_{s-\tau})$ for $s = 1+\tau,\ldots,t$. Thus, restricting γ_s this way ensures that we can evaluate $x^*_{s-\tau} \equiv x_{\gamma_s-\tau}$. Setting $x_s(t) = 1$ for all s implies this restriction is not needed. Similarly, we also restrict the support of η_s to the set $\{R+\tau, ..., T+\tau\}$. This ensures that $\eta_s - \tau$ is in the set $\{R,\ldots,T\}$, for which we have both final and preliminary values of the variables. This is particularly important in step 3, where we need to obtain the predictors $x^*_t(t) = x_{\eta_{t+\tau}-\tau}(\eta_{t+\tau}-\tau)$.*

Lemma 5.1 *Under Assumptions 1-5 and letting $l \to \infty$ such that $l/\min\{\sqrt{R},\sqrt{P}\} \to 0$,*

$$\tilde{S}^*_P \equiv P^{-1/2}\sum_{t=R}^{T}\left(f^*_{t+\tau|r'}(\hat{\beta}^*_t) - f_{t+\tau|r'}(\bar{\beta}_t)\right) = S^*_{1P} + FBS^*_{2P} + o^*_p(1),$$

where

$$S^*_{1P} = P^{-1/2}\sum_{t=R}^{T}\left(f^*_{t+\tau|r'} - f_{t+\tau|r'}\right),$$

and

$$S^*_{2P} = a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h^*_s - \bar{h}_R) + P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(h^*_{R+i} - \bar{h}_P) \equiv S^*_{2P.1} + S^*_{2P.2},$$

where $h^*_t = x^*_{t-\tau}(y^*_t - x^{*\prime}_{t-\tau}\beta_0)$, $\bar{h}_R = R^{-1}\sum_{s=1+\tau}^{R} h_s$ and $\bar{h}_P = P^{-1}\sum_{s=R+\tau}^{T+\tau} h_s$.

Under our assumptions, $S^*_{1P} \xrightarrow{d} N(0,\Omega_1)$ by Fitzenberger (1998) (cf. Theorem 3.1). As in the simple location model, the term S^*_{2P} has two components, $S^*_{2P.1}$ and $S^*_{2P.2}$, both centered at zero asymptotically. To see this, note that $\bar{h}_R = E^*\left(R^{-1}\sum_{s=1+\tau}^{R} h^*_s\right) + O_P(l/R)$, whereas $\bar{h}_P = E^*\left(P^{-1}\sum_{i=1}^{P-1} h^*_{R+i}\right) + O_P(l/P)$. In addition, we can show that this term's bootstrap variance converges to $\Omega_2 = \Omega_{2.1}+\Omega_{2.2}$. Since the bootstrap covariance between $S^*_{2P.1}$ and S^*_{1P} is zero by construction and we show that the covariance between S^*_{1P} and $S^*_{2P.2}$ is asymptotically equal to Ω_{12}, the following result follows.

Theorem 5.1 *Suppose Assumptions 1-5 hold and $l \to \infty$ such that $l/\min\{\sqrt{R},\sqrt{P}\} \to 0$. Then,*

$$\sup_{u\in\mathbb{R}}\left|P^*\left(\tilde{S}^*_P \leq u\right) - \Pr\left(\tilde{S}^\mu_P \leq u\right)\right| \to_p 0,$$

where $\tilde{S}^\mu_P = \tilde{S}_P - P^{1/2}E\left(f_{t+\tau|r'}\right)$.

The proof of Theorem 5.1 relies in part on Lemma A.4 in Appendix A (which shows the consistency of the bootstrap variance estimator of $\tilde{S}^*_P$). Theorem 5.1 implies that our bootstrap algorithm can be used to approximate the asymptotic distribution of $\tilde{S}_P^{\mu}$, a centered version of $\tilde{S}_P$. When the null hypothesis $H_0 : E\left(f_{t+\tau|r'}\right) = 0$ is true, $\tilde{S}_P^{\mu}$ coincides with the test statistic $\tilde{S}_P$, in which case Theorem 5.1 proves the consistency of the bootstrap critical values obtained from $\tilde{S}_P^*$. When the null hypothesis does not hold, Theorem 5.1 shows that the bootstrap distribution of $\tilde{S}_P^*$ is consistent for the distribution of $\tilde{S}_P^{\mu}$, implying that the bootstrap test based on $\tilde{S}_P^*$ has power.

5.3 Bootstrap results for nested linear models

The bootstrap algorithm in the previous section can be applied when conducting a test of equal predictability between nested models. Nevertheless, when the models are nested, the algorithm can be simplified considerably. In the following we present that simplification in the context of tests of equal accuracy under quadratic loss. By doing so we are also able to discuss how the properties of the data revisions affect whether Ω is positive.

In this application the loss differential defines the function $f_{t+\tau|r'}$ and takes the form

$$f_{t+\tau|r'} \equiv f\left(y_{t+\tau|r'}, x_t\left(t\right), \beta_0\right) = \left(y_{t+\tau|r'} - x_{1,t}'\left(t\right)\beta_{1,0}\right)^2 - \left(y_{t+\tau|r'} - x_{2,t}'\left(t\right)\beta_{2,0}\right)^2,$$

where $x_{2,t}\left(t\right) = \left(x_{1,t}\left(t\right)', x_{22,t}\left(t\right)'\right)'$ and $\beta_0 = \left(\beta_{1,0}', \beta_{2,0}'\right)'$. Under the null of equal predictive ability, model 2 includes $\dim(x_{22,s}(t)) = k_{22}$ excess parameters, i.e., $\beta_{2,0} = (\beta_{1,0}', 0')'$ and $x_{1,t}(t)'\beta_{1,0} = x_{2,t}(t)'\beta_{2,0}$.

As we have done before, we let $\hat{\beta}_t = \left(\hat{\beta}_{1,t}', \hat{\beta}_{2,t}'\right)'$ denote the estimators of β_0 based on final data and we let $\hat{\beta}\left(t\right) = \left(\hat{\beta}_1'\left(t\right), \hat{\beta}_2'\left(t\right)\right)'$ denote their real-time data analogs. $\hat{S}_P$ is the test statistic based on $\hat{\beta}\left(t\right)$, and $\tilde{S}_P$ denotes its analog based on $\hat{\beta}_t$. Lemma 4.1 immediately implies that

$$\hat{S}_P = P^{-1/2}\sum_{t=R+1}^{T}\left(\left(y_{t+\tau|r'} - x_{1,t}'\left(t\right)\hat{\beta}_{1,t}\right)^2 - \left(y_{t+\tau|r'} - x_{2,t}'\left(t\right)\hat{\beta}_{2,t}\right)^2\right) + o_p(1) \equiv \tilde{S}_P + o_p\left(1\right). \quad (3)$$

More importantly, since the models are nested we know that under the null, not only is $Ef_{t+\tau|r'} = 0$, but also $f_{t+\tau|r'} = 0$ since $x_{1,t}(t)'\beta_{1,0} = x_{2,t}(t)'\beta_{2,0}$. This makes bootstrapping the distributions of $\hat{S}_P$ and $\tilde{S}_P$ easier since $\tilde{S}_P = FBP^{-1/2}\sum_{t=R}^{T}H(t) + o_p(1)$ and the uncertainty in this term is determined solely by fully revised data, and hence we no longer need to replicate the triangular structure of the different vintages.

Before delineating this bootstrap, it is useful to note that the expansion for $\tilde{S}_P$ simplifies even further under the null hypothesis. Let $F \equiv E[\frac{\partial}{\partial\beta'}f_{t+\tau|r'}\left(\beta\right)] = [F_1, F_2]$, with $F_i \equiv E[\frac{\partial}{\partial\beta_i'}f_{t+\tau|r'}\left(\beta\right)]$ for $i = 1, 2$ and recall that $B = diag(B_i)$. Since the models are nested we know that for a selection matrix $J' = (I_{k_1\times k_1}, 0_{k_1\times k_{22}})$, $H_1(t) = J'H_2(t)$ and $F_1 = -F_2J$ and hence

$$\tilde{S}_P = F_2(-JB_1J' + B_2)P^{-1/2}\sum_{t=R}^{T}H_2(t) + o_p\left(1\right). \quad (4)$$

In addition, noting that for $t = R, \ldots, T$,

$$\hat{\beta}_{2,t} - \beta_{2,0} = B_2\left(t\right) H_2\left(t\right),$$

we can further rewrite (4) as

$$\tilde{S}_P = F_2(-JB_1J'B_2^{-1} + I_{k_2})P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_{2,t} - \beta_{2,0}) + o_p\left(1\right).$$

This expansion shows that we can replicate the distribution of $\tilde{S}_P$ by replicating the distribution of $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_{2,t} - \beta_{2,0})$. Since this term only depends on OLS estimates from the larger model evaluated with final data, we can rely on Corradi and Swanson's (2007) method to bootstrap its distribution. This combined with a consistent estimator of the factor $F_2(-JB_1J'B_2^{-1} + I_{k_2})$ provides a valid bootstrap method for computing the quantiles of $\tilde{S}_P$ under the null hypothesis.

The bootstrap algorithm is as follows.

Bootstrap algorithm for nested linear models

1. Let $T-(1+\tau)+1 = kl$ and generate $I_1, \ldots, I_k \sim$ i.i.d. Uniform on $\{1+\tau, \ldots, T-l+1\}$. Then, for each $i = 1, \ldots, k$ and $j = 1, \ldots, l$, set $I_i + (j-1) = \gamma_{1+\tau+(i-1)l+(j-1)}$ and let

$$\{\gamma_s : s = 1+\tau, \ldots, T\} = \{\gamma_{1+\tau+(i-1)l+(j-1)} : i = 1, \ldots, k;\ j = 1, \ldots, l\}.$$

2. For $s = 1+\tau, \ldots, T$, set

$$z_s^* \equiv (y_s^*, x_{2,s-\tau}^{*\prime})' = (y_{\gamma_s}, x'_{2,\gamma_s-\tau})',$$

and for $t = R, \ldots, T$, compute

$$\tilde{\beta}_{2,t}^* = \arg\min_{\beta_2}\left[t^{-1}\sum_{s=1+\tau}^{t}(y_s^* - {x_{2,s-\tau}^*}'\beta_2)^2 - \beta_2'(T-\tau)^{-1}\sum_{s=1+\tau}^{T}2(y_s - x'_{2,s-\tau}\hat{\beta}_{2,t})x_{2,s-\tau}\right].$$

3. Compute

$$\tilde{S}_P^* = \hat{F}_2(-J\hat{B}_1J'\hat{B}_2^{-1} + I_{k_2})P^{-1/2}\sum_{t=R}^{T}(\tilde{\beta}_{2,t}^* - \hat{\beta}_{2,t}),$$

where $\hat{F}_2 = 2P^{-1}\sum_{t=R}^{T}(y_{t+\tau|r'} - x'_{2,t}\left(t\right)\hat{\beta}_{2,T})x'_{2,t}(t)$, and $\hat{B}_i = (T^{-1}\sum_{s=1}^{T}x_{i,s}x'_{i,s})^{-1}$ are consistent estimates of F_2 and B_i for $i = 1, 2$, respectively.

Steps 1 and 2 amount to using the block bootstrap method of Corradi and Swanson (2007) to replicate the distribution of $P^{-1/2}\sum_{t=R}^{T}(\hat{\beta}_{2,t} - \beta_{2,0})$. Contrary to our previous bootstrap algorithms, we need only one set of MBB indices in Step 1. The main reason for using two sets of MBB indices in Step 1 of our previous methods was the need to replicate the triangular structure of the vintages data. This is no longer required because the term that depends on the function $f_{t+\tau|r'}$ is zero when the models are nested. The other key difference is that we now incorporate a bias correction term in the definition of $\tilde{\beta}^*_{2,t}$, i.e. we do not estimate $\beta_{2,0}$ using the standard OLS estimator. This correction term is one way of correcting for the bias introduced by the recursive estimation scheme in the bootstrap world and was suggested by Corradi and Swanson (2007) in a context without data revisions.

The asymptotic validity of $\tilde{S}^*_P$ follows from Theorem 1 of Corradi and Swanson (2007) and the consistency of $\hat{F}_2$, $\hat{B}_i$ for $i = 1, 2$, provided the condition $F_2(-JB_1J' + B_2) \neq 0$ holds (ensuring that Ω is positive). This last condition is non-trivial and can depend on the statistical properties of the revision process. Following Mankiw, Runkle, and Shapiro (1984), we treat revisions as consisting of news (v) and noise (w) components. A revision is said to be pure news if it is uncorrelated with any data available at the time of the provisional estimate. If the revision is correlated with the provisional estimate, then the revision contains a noise component.

Specifically, in the context of verifying whether $F_2(-JB_1J' + B_2) \neq 0$, consider the following example in which $r = 2$ and an $AR(2)$ model nests an $AR(1)$ model, so that $x_{2,t} = (1, y_t, y_{t-1})'$ and $x_{1,t} = (1, y_t)'$. The fully revised data and initial releases then take the form

$$
\begin{aligned}
y_t &= \delta_0 + \delta_1 y_{t-1} + v_t + e_t, \\
y_t(t) &= y_t - v_t + w_t,
\end{aligned}
$$

for error terms e_t, and revision components v_t and w_t, which are mutually independent i.i.d. zero-mean Gaussian variates with variances σ^2_e, σ^2_v, and σ^2_w, respectively. If $\tau = 1$ and the initial release is used for forecast evaluation, this implies that F_2 takes the form

$$
\begin{aligned}
F_2 &= 2E\left[(y_{t+1|1} - x'_{2,t}(t)\,\beta_2)x'_{2,t}(t)\right] \\
&= 2E\left[(y_{t+1} - v_{t+1} + w_{t+1} - \delta_0 - \delta_1(y_t - v_t + w_t))(1, y_t - v_t + w_t, y_{t-1})\right] \\
&= 2E\left[(e_{t+1} + w_{t+1} - \delta_1(-v_t + w_t))(1, y_t - v_t + w_t, y_{t-1})\right] \\
&= (0, -2\delta_1\sigma^2_w, 0).
\end{aligned}
$$

We immediately observe a simple instance for which F_2, and hence $F_2(-JB_1J' + B_2)$, will be zero. F_2 is zero if δ_1 is zero, or if there is no noise component to the revision (i.e., $\sigma^2_w = 0$).

6 Monte Carlo simulations

In this section, we consider the finite sample size and power of bootstrap-based inference for tests of equal predictive ability when data are subject to revisions. In each case we use OLS to estimate two predictive models $x'_{i,t}\beta_i$ $i = 1, 2$ and evaluate accuracy under quadratic loss. The design of the experiments is comparable to that in Clark and McCracken (2009).

6.1 Non-nested models

We begin with size and power experiments associated with tests of equal predictive ability between two non-nested models. The final data are generated according to

$$y_t = 0.4x_{1,t-1} + (0.4 + \Delta)x_{2,t-1} + e_{y,t} + v_{y,t},$$

where $e_{y,t}$ and $v_{y,t}$ are independently generated as i.i.d. Gaussian variables with mean zero and variance equal to $\sigma^2_{e,y}$ and $\sigma^2_{v,y}$, respectively. Similarly, we let each regressor's final data be generated as $x_{i,t} = e_{x_i,t} + v_{x_i,t},\ i = 1, 2$, where $e_{x_i,t}$ and $v_{x_i,t}$ are also mutually independent (jointly with $e_{y,t}$ and $v_{y,t}$) i.i.d. Gaussian random variables with mean zero and variances $\sigma^2_{e,x}$ and $\sigma^2_{v,x}$, respectively. For instance, we can think of y_t as the quarterly inflation and $x_{1,t-1}$ and $x_{2,t-1}$ as two different measures of economic activity, as in the empirical application considered by Clark and McCracken (2009) and the related application considered later in Section 7. When $\Delta = 0$, these two measures have the same predictive content for inflation, but not otherwise.

We consider the case of a single revision, where time t's preliminary estimates of y_t and $x_{i,t}$ are

$$y_t\,(t) = y_t - v_{y,t} + w_{y,t}, \quad \text{and} \quad x_{i,t}\,(t) = x_{i,t} - v_{x_i,t} + w_{x_i,t} \text{ for } i = 1, 2,$$

with $w_{y,t}$ and $w_{x_i,t}$ denoting i.i.d. Gaussian random variables with variances $\sigma^2_{w,y}$ and $\sigma^2_{w,x}$, respectively. These random variables are also mutually independent (jointly with $e_{y,t}$, $v_{y,t}$, $e_{x_i,t}$ and $v_{x_i,t}$). Following the real-time data literature, we interpret $v_{y,t}$ and $v_{x_i,t}$ as the news components of the revisions to $y_t\,(t)$ and $x_i\,(t)$, respectively, whereas $w_{y,t}$ and $w_{x_i,t}$ represent the noise components.[8]

Our goal is to test the equal predictability of a forecast of $y_{t+1}\,(t+1) \equiv y_{t+1|1}$ using two non-nested models, each based on a real-time predictor $x_{i,t}\,(t)$ for $i = 1, 2$. The null hypothesis is then

$$H_0 : E(f_{t+1|1}) \equiv E\Big[(y_{t+1|1} - x_{1,t}\,(t)\,\beta_{1,0})^2 - (y_{t+1|1} - x_{2,t}\,(t)\,\beta_{2,0})^2\Big] = 0.$$

This is true when $\Delta = 0$ (used in the experiments describing the size properties of our test), but not otherwise (we set $\Delta = 0.6$ when evaluating power).

We follow Clark and McCracken (2009) and consider two different data-generating processes, DGP1 and DGP2. These include both noise and news components in the revisions and are described as

[8] To explain briefly this terminology, take as an example the equations that describe y_t and $y_t\,(t)$. Given that $v_{y,t}$ enters y_t and $y_t\,(t)$ with a positive and negative sign, respectively, the preliminary value $y_t\,(t)$ does not depend on the component $v_{y,t}$, implying that this term is uncorrelated with $y_t\,(t)$. This is the sense in which $v_{y,t}$ describes "news". Instead, the presence of $w_{y,t}$ in the preliminary value $y_t\,(t)$ and its absence from y_t explain why we call this component a "noise" component.

"DGP1, noise and news" and "DGP2, noise and news" in Table 3 below. We also consider two variations of these DGPs, without a noise component, i.e. we set $\sigma^2_{w,y} = \sigma^2_{w,x} = 0$. In this case, we write "DGP1, news only" and "DGP2, news only," respectively.

The versions of DGP1 and DGP2 that contain both noise and news differ in the way they parametrize the revisions process. More specifically,

- DGP1: $\sigma^2_{e,y} = 0.1,\ \sigma^2_{v,y} = 0.9,\ \sigma^2_{w,y} = 0.2,\ \sigma^2_{e,x} = 1.7,\ \sigma^2_{v,x} = 0.3,\ \sigma^2_{w,x} = 2.$
- DGP2: $\sigma^2_{e,y} = 0.8,\ \sigma^2_{v,y} = 0.2,\ \sigma^2_{w,y} = 0.2,\ \sigma^2_{e,x} = 1.7,\ \sigma^2_{v,x} = 0.3,\ \sigma^2_{w,x} = 0.5.$

As discussed by Clark and McCracken (2009), DGP2 is motivated by results in Aruoba (2008), which provides empirically relevant values for the variance parameters based on real-time U.S. data on the change in inflation and the output gap over the period 1965-2003. For instance, these parameter values imply a correlation of about -0.2 between the revision in y (given by $y_t - y_t(t)$) and its initial estimate $y_t(t)$, in line with the data. The variance of the revision in y_t is given by $\sigma^2_{v,y} + \sigma^2_{w,y} = 0.2 + 0.2 = 0.4$ and is about 25% of the total variance of y_t . Similarly, the variance of the revisions for each x variables (given by $\sigma^2_{v,x} + \sigma^2_{w,x} = 0.8$) is about 40% of the total variance of its final value.

The biggest difference between DGP1 and DGP2 is that the revision variance of y is much larger in DGP1 and is now about 70% of the total variance in y. Similarly, the revision variance of each x variable is 15% larger than the variance of the final series. These differences imply that the impact of the revision process is larger for DGP1 than for DGP2 and can lead to substantial differences in actual power, as we will see when discussing Table 3 below.

At each forecast origin, we use the generated real-time data to make a one-step-ahead forecast for the target variable $y_{t+1|1}$. Forecasts take the form $x'_{i,t}(t)\hat{\beta}_i(t)$ with subsequent forecast errors $\hat{u}_{i,t+1|1} = y_{t+1|1} - x_{i,t}(t)'\hat{\beta}_i(t)$ for $i = 1, 2$. As noted above, accuracy is evaluated under quadratic loss.

Table 3 contains the results. We consider three different tests. One is the bootstrap test described in Section 5.2, which is used to obtain bootstrap critical values for $\hat{S}_P$. This test is a percentile-type test that does not require studentizing the statistic $\hat{S}_P$. It is labeled "Bootstrap" in Table 3. We also include two alternative tests. These tests are based on studentized versions of $\hat{S}_P$ and rely on critical values taken from the standard normal distribution (hence, they do not involve the bootstrap and are included as benchmark methods). One is the Diebold and Mariano (1995) test, labeled $t(\hat{\Omega}_{1P})$ in Table 3. It takes the form $t(\hat{\Omega}_{1P}) = \hat{\Omega}_{1P}^{-1/2}\hat{S}_P$, where $\hat{\Omega}_{1P}$ is a consistent estimator of Ω_1 given in (1). The third test (which appears under the label $t(\hat{\Omega}_P)$ in Table 3) is the one proposed by Clark and McCracken (2009). It is given by $t(\hat{\Omega}_P) = \hat{\Omega}_P^{-1/2}\hat{S}_P$, where $\hat{\Omega}_P$ is a consistent estimator of Ω_P in (1). See Section 3.3 of Clark and McCracken (2009) for the details in obtaining $\hat{\Omega}_{1P}$ and $\hat{\Omega}_P$. Relative to their simulations, the only difference is that the relevant long-run variances are estimated using a bandwidth of $\lfloor \min\{R^{1/3}, P^{1/3}\} \rfloor$ rather than an ad hoc rule of twice the forecast horizon. Results are obtained with 10,000 Monte Carlo replications and 499 bootstrap replications each. We set $R = 80$ and allow P to grow from 20 to 160. The block length is equal to $l = \lfloor \min\{R^{1/3}, P^{1/3}\} \rfloor$. This choice ensures that $l \to \infty$ such that $l/\min\{\sqrt{R}, \sqrt{P}\} \to 0$, as assumed in Theorem 5.1. It also ensures that $l > 1$ when $P = 20$ and $R = 80$. The nominal level α is 5%.

The left panel of Table 3 shows that the asymptotic-based tests are typically oversized, particularly when P is small. This is especially true for the Diebold and Mariano test, which does not account for parameter estimation uncertainty. The Clark and McCracken test, which accounts for the presence of parameter estimation error, is usually more accurately sized. However, the results in Table 3 suggest that $t(\hat{\Omega}_P)$ still tends to overreject in finite samples. This is especially true for the DGPs without noise, where the rejections rates of $t(\hat{\Omega}_P)$ vary between 0.090 ($P = 20$) and 0.057 ($P = 160$) for DGP1, news only. These numbers are 0.085 and 0.057 for DGP2, news only, respectively. In contrast, the bootstrap test yields rejection rates equal to 0.062 and 0.051, for DGP1, news only, when $P = 20$ and $P = 160$, respectively. These rates are 0.055 and 0.050 for DGP2, news only. Hence, the bootstrap largely corrects the size distortions of the asymptotic-based tests when the DGP does not contain noise. This is also true for the DGPs that include noise and news in the revision process. In this case, Table 3 shows that the bootstrap can be slightly conservative, especially for DGP1. For DGP2, with noise and news, the degree of conservativeness of the bootstrap is smaller than for DGP1. Overall, it appears that the bootstrap test does a reasonable job controlling size in finite samples and, more importantly, does so without having to compute $\hat{\Omega}_P$.

The right panel of Table 3 shows that the three test statistics have power converging to 1 as P increases for all DGPs, except DGP1 with noise and news in the revision process (we provide an explanation below). The results show that the bootstrap rejection rates are typically smaller than those of the asymptotic-based tests under the alternative. This is not surprising since the latter have larger rejection rates than the bootstrap test even under the null hypothesis.

Unlike the case where revisions are pure news, actual power of the test can be quite low when revisions also have a noise component. One example is "DGP1, noise and news." While not immediately obvious, the root of the problem lies with how the noisy revisions affect the mean loss differential. For example, in DGPs 1 and 2 the population squared forecast errors take the form $u^2_{i,t+1|1} = (y_{t+1|1} - x_{i,t}(t)\beta_{i,0})^2$, $i = 1, 2$. After taking expectations, straightforward algebra reveals

$$E(u^2_{i,t+1|1}) = \beta^2_{i,0}\sigma^2_{e,x} + \beta^2_{i,0}\sigma^2_{v,x} + \sigma^2_{e,y} + \sigma^2_{w,y} + \beta^2_{j,0}\sigma^2_{v,x} + \beta^2_{j,0}\sigma^2_{w,x},$$

Table 3: Non-nested model size and power results with 5% nominal level

Tests	P = 20	40	80	160	P = 20	40	80	160
	size: DGP1, news only				power: DGP1, news only			
$t(\hat{\Omega}_{1P})$	0.110	0.084	0.074	0.064	0.701	0.937	0.999	1.000
$t(\hat{\Omega}_{P})$	0.090	0.071	0.064	0.057	0.695	0.935	0.999	1.000
Bootstrap	0.062	0.057	0.052	0.051	0.591	0.893	0.996	1.000
	size: DGP2, news only				power: DGP2, news only			
$t(\hat{\Omega}_{1P})$	0.100	0.085	0.075	0.060	0.492	0.742	0.960	0.999
$t(\hat{\Omega}_{P})$	0.085	0.075	0.068	0.058	0.485	0.742	0.961	1.000
Bootstrap	0.055	0.053	0.056	0.050	0.400	0.676	0.941	0.999
	size: DGP1, noise and news				power: DGP1, noise and news			
$t(\hat{\Omega}_{1P})$	0.122	0.109	0.119	0.133	0.113	0.098	0.104	0.119
$t(\hat{\Omega}_{P})$	0.062	0.047	0.046	0.046	0.084	0.068	0.068	0.070
Bootstrap	0.034	0.030	0.031	0.035	0.056	0.054	0.061	0.067
	size: DGP2, noise and news				power: DGP2, noise and news			
$t(\hat{\Omega}_{1P})$	0.101	0.079	0.065	0.048	0.230	0.348	0.550	0.830
$t(\hat{\Omega}_{P})$	0.083	0.069	0.061	0.050	0.223	0.347	0.557	0.842
Bootstrap	0.051	0.048	0.050	0.043	0.164	0.292	0.513	0.817

for $i \neq j$ and hence

$$E(u_{1,t+1|1}^2 - u_{2,t+1|1}^2) = (\beta_{2,0}^2 - \beta_{1,0}^2)(\sigma_{e,x}^2 - \sigma_{w,x}^2).$$

When the revisions are pure news, as in DGP1 and 2, "news only," this expectation is non zero so long as $\beta_{1,0}$ and $\beta_{2,0}$ differ since $\sigma_{e,x}^2 > 0$ and $\sigma_{w,x}^2 = 0$. However, in the presence of a noise component, whether it is nonzero also depends on the relative magnitudes of $\sigma_{e,x}^2$ and $\sigma_{w,x}^2$. In fact, if we use the specific parameterization of DGP1, we find that the absolute expected loss differential takes the value 1.43 when the data revision consists of news only, but reduces to 0.25 when the data revision consists of news and noise. Clearly, when the data revision consists of news and noise, the mean loss differentials are lower in absolute value, which in turn leads to significant reductions in power.

6.2 Nested models

For the nested case, we consider two DGPs with both noise and news. More specifically, the final data are generated according to

$$y_t = 0.7y_{t-1} + \beta_{22}x_{t-1} + e_{y,t} + v_{y,t}$$

$$x_t = 0.7x_{t-1} + e_{x,t} + v_{x,t}$$

$$Var\begin{pmatrix} e_{y,t} \\ e_{x,t} \\ v_{y,t} \\ v_{x,t} \end{pmatrix} = \begin{pmatrix} 0.8 & cov(e_{y,t}, e_{x,t}) & 0 & 0 \\ cov(e_{y,t}, e_{x,t}) & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}$$

Table 4: Nested model size and power results with 5% nominal level

Tests	P = 20	40	80	160	P = 20	40	80	160
	size: DGP1, noise and news				power: DGP1, noise and news			
$t(\hat{\Omega}_{1P})$	0.101	0.122	0.170	0.248	0.290	0.413	0.637	0.873
$t(\hat{\Omega}_{P})$	0.102	0.086	0.079	0.066	0.628	0.732	0.842	0.941
Bootstrap	0.095	0.080	0.073	0.061	0.619	0.726	0.837	0.939
	size: DGP2, noise and news				power : DGP2, noise and news			
$t(\hat{\Omega}_{1P})$	0.044	0.027	0.019	0.012	0.083	0.073	0.070	0.081
$t(\hat{\Omega}_{P})$	0.111	0.100	0.085	0.074	0.376	0.386	0.385	0.379
Bootstrap	0.105	0.095	0.083	0.072	0.372	0.382	0.383	0.378

where $cov(e_{y,t}, e_{x,t}) = 0.35$ for DGP1 and $cov(e_{y,t}, e_{x,t}) = 0.25$ for DGP2. We set $\beta_{22} = 0$ when evaluating size and $\beta_{22} = 0.3$ when evaluating power.

The structure of the revisions is similar to what we considered in the non-nested case, i.e. we consider a single revision, given by

$$y_t\,(t) = y_t - v_{y,t} + w_{y,t} \text{ and } x_t\,(t) = x_t - v_{x,t} + w_{x,t},$$

where $w_{y,t}$ and $w_{x,t}$ are the noise components. They are generated as i.i.d. Gaussian random variables with mean zero and variances that differ across DGPs. In DGP1, we set $\sigma^2_{w,y} = 1.8$ and $\sigma^2_{w,x} = 0.5$. As explained by Clark and McCracken (2009), this implies a correlation between the revision in y and its final estimate of about -0.7, and a revisions variance about the same as the variance of the final value, which is larger than what is observed for real-time data in inflation. With $cov(e_{y,t}, e_{x,t}) = 0.35$, this choice of parameters implies a large value of Ω. In DGP2, we set $\sigma^2_{w,y} = 0.2$ and $\sigma^2_{w,x} = 0.5$. This choice of parameters implies more realistic values for the correlation between the revisions and the final values (as well as their proportional variances), but lead to smaller values of Ω. As we will see later, this distinction seems to play a role in the finite-sample efficacy of the asymptotics.

At each forecast origin, we use the generated real-time data to make a one-step-ahead forecast for the target variable $y_{t+1|1}$. In each DGP the forecast takes the form $x'_{i,t}(t)\hat{\beta}_i(t)$ for $i = 1, 2$, where $x_{1,t}(t)' = y_t(t)$ and $x_{2,t}(t)' = (y_t(t), x_t(t))$. The forecast errors take the form $\hat{u}_{i,t+1|1} = y_{t+1|1} - x_{i,t}(t)'\hat{\beta}_i(t)$ for $i = 1, 2$. For all DGPs, we test equal predictive ability under quadratic loss. In contrast to the experiments in Table 3, here we use the simpler bootstrap algorithm described in Section 5.3 to obtain critical values for the test statistic $\hat{S}_P$. We set the block length to $l = \lfloor T^{1/5} \rfloor$, which satisfies the block length requirement of Theorem 1 in Corradi and Swanson (2007). We include the same two asymptotic tests, $t(\hat{\Omega}_{1P})$ and $t(\hat{\Omega}_P)$, as benchmarks. These tests are defined as previously, and their critical values are obtained from the standard normal distribution.

Table 4 contains results for the nested models comparisons. The results replicate those of Clark and

McCracken (2009). Specifically, for DGP1, where Ω is large, the asymptotic test based on the Diebold and Mariano statistic $t(\hat{\Omega}_{1P})$ is oversized under the null hypothesis, with null rejections rates that increase from 0.101 when $P = 20$ to 0.248 when $P = 160$. In sharp contrast, in DGP2 the Diebold-Mariano version is severely undersized with rejection rates that decline from 0.044 when $P = 20$ to 0.012 when $P = 160$. The bootstrap test is comparable to the Clark and McCracken test and both lead to better size control under the null hypothesis.

6.3 Robustness

Each of the simulations were designed to align with our assumptions. For example, we require r to be finite and, in particular, to be small relative to both R and P. In addition, we abstract from annual benchmark revisions. In the following we provide a limited collection of simulations designed to highlight the performance of the bootstrap when these assumptions are relaxed.

6.3.1 Small number of revisions

Here we consider a modified version of DGP2 applied to tests of equal accuracy for non-nested models. The environment is largely the same as that in Section 6.1 except that we now allow values of r equaling $4, 8, 12$, and 16. For example, the final, first, and intermediate releases $j \in \{1, \ldots, r-1\}$ of the dependent variable take the form

$$\begin{aligned} y_{t|r} &= 0.4x_{1,t-1} + (0.4 + \Delta)x_{2,t-1} + e_{y,t} + \sum_{i=1}^{r-1} v_{y,t|i}, \\ y_{t|1} &= 0.4x_{1,t-1} + (0.4 + \Delta)x_{2,t-1} + e_{y,t} + \sum_{i=1}^{r-1} w_{y,t|i}, \\ y_{t|(j+1)} &= y_{t|j} - w_{y,t|j} + v_{y,t|j}. \end{aligned}$$

Similarly, the final, first, and intermediate releases of each regressor $x_{i,t}$ $i = 1, 2$ take the form

$$\begin{aligned} x_{i,t|r} &= e_{x_i,t} + \sum_{l=1}^{r-1} v_{x_i,t|l}, \\ x_{i,t|1} &= e_{x_i,t} + \sum_{l=1}^{r-1} w_{x_i,t|l}, \\ x_{i,t|(j+1)} &= x_{i,t|j} - w_{x_i,t|j} + v_{x_i,t|j}. \end{aligned}$$

While the parameterization is similar to DGP2, we rescale the variance of the news and noise components so that their contribution to the variances of y and x_i are invariant to the choice of r.

- DGP 3: $\sigma^2_{e,y} = 0.8$, $\sigma^2_{v,y} = 0.2/(r-1)$, $\sigma^2_{w,y} = 0.2/(r-1)$, $\sigma^2_{e,x} = 1.7$, $\sigma^2_{v,x} = 0.3/(r-1)$, $\sigma^2_{w,x} = 0.5/(r-1)$.

Table 5: Non-nested model size and power results with 5% nominal level: multiple revisions

r =2	4	8	12	16	r =2	4	8	12	16
size: DGP3, news only					power: DGP3, news only				
0.053	0.050	0.053	0.053	0.049	0.676	0.688	0.692	0.686	0.683
size: DGP3, noise and news					power: DGP3, noise and news				
0.048	0.049	0.047	0.041	0.043	0.292	0.285	0.293	0.289	0.290

In Table 5 we report actual size and power of our bootstrap test for a range of values of r when the sample sizes are $R = 80$ and $P = 40$. As we saw in Table 3, regardless of the choice of r, actual size of the test tends to be reasonable when using the percentile bootstrap. In addition, while actual power varies across DGPs, the choice of r does not seem to be a leading determinant of that variation.

6.3.2 Annual revisions

Here we consider a stylized example of how our bootstrap can, but need not, be robust to the presence of annual revisions. Suppose that in each quarterly vintage t there is an initial release $y_{t|1}$. However, revisions occur only once a year and when they do, all previous releases are final. Assume that all realizations y_s, $s = 1, \ldots, R-1$ are final.

We consider two autoregressive models for forecasting y_{t+1}, one based on an AR(1) model, which uses one lagged value as a predictor, and another based on a restricted version of an AR(2) model (where the twice lagged value is the predictor). Specifically, the DGP can be described as

$$y_t = x_{t-1}\beta + e_t + v_t \quad \text{and} \quad y_{t|1} = y_t - v_t + w_t,$$

where $e_t \sim$ i.i.d.$N(0, \sigma_e^2)$, $v_t \sim$ i.i.d.$N(0, \sigma_v^2)$, and $w_t \sim$ i.i.d.$N(\mu_w, \sigma_w^2)$. We set $x_{t-1} = y_{t-1}$ for the AR(1) model and $x_{t-1} = y_{t-2}$ for the AR(2) model. Under the null hypothesis, $\beta = 0$, which implies $y_t \sim$ i.i.d.$N(0, \sigma^2)$, with $\sigma^2 = \sigma_e^2 + \sigma_v^2$. We set $\beta = 0.5$ under the alternative hypothesis.

We consider a test of zero-mean prediction error based on an OLS-estimated autoregressive model that does not contain an intercept. The one-step-ahead forecast is evaluated using the fully revised value $y_{t+1|r'} = y_{t+1}$. The moment of interest takes the form $E(f_{t+1}) = E(y_{t+1} - x_t(t)\beta) = 0$ where the specific form of $x_t(t)$ depends on the lag length of the model.

For the AR(1) model, $x_t(t) = y_{t|1}$ for all t. Given our DGP, $\beta = 0$, which implies $f_{t+1} = y_{t+1}$ and the null hypothesis holds. Since the revision process is finite lived and the functional form for $f_{t+1}(\beta)$ is time invariant, the asymptotics in Clark and McCracken still apply. Intuitively, our bootstrap algorithm will also apply because it will enforce the feature that for every vintage, the most recent value will be an initial release while all previous values will be final. Put differently, while our algorithm

Table 6: Size and power results: AR(1) and AR(2) models

λ =1	4	12	λ =1	4	12
size: AR(1)			power: AR(1)		
0.056	0.051	0.072	0.959	0.964	0.977
size: AR(2)			power: AR(2)		
0.056	0.111	0.163	0.061	0.944	0.988

does not replicate the entire pattern of the data when annual revisions are present, it replicates what is needed for this example.

Now suppose the model has a twice-lagged value (as in the restricted AR(2) model described above) and hence $x_t(t) = y_{t-1}(t)$. We immediately find that the functional form for f_{t+1} changes across the calendar year. In most periods it takes the form $f_{t+1}^{(1)}(\beta) = y_{t+1} - \beta y_{t-1|1}$, but during the annual revision it takes the form $f_{t+1}^{(2)}(\beta) = y_{t+1} - \beta y_{t-1}$. In both cases, under our assumed DGP, $f_{t+1} = y_{t+1}$ and the null hypothesis holds for all t. The problem is that $Ef_{t+1,\beta}^{(j)}$ need not be constant for all t. If we let $F^{(j)}$ denote $Ef_{t+1,\beta}^{(j)}$ then $F^{(1)} = -Ey_{t-1|1}$ and $F^{(2)} = -Ey_{t-1} = 0$, which are distinct so long as $F^{(1)}$ is non-zero. Since $F^{(j)}$ varies, the asymptotics in Clark and McCracken and our bootstrap algorithm do not apply, because neither is designed to distinguish between regular vintages and vintages that contain annual revisions.

In Table 6 we provide simulation evidence on the actual size and power of the bootstrap-based test of zero-mean prediction error in the presence of an annual revision. For both models, we let

$$\sigma_e^2 = 0.3, \quad \sigma_v^2 = 0.2, \quad \sigma_w^2 = 0.2, \quad \mu_w = 0.85.$$

The initial estimation sample size is $R = 80$, while the out-of-sample size is $P = 80$. We set the bootstrap's block length to 1 since our examples have the m.d.s. property. We consider annual revision frequencies of $\lambda = 1,\ 4,\ 12$. When $\lambda = 1$, the vintages have a single regular revision structure. When $\lambda > 1$, revisions only arise with frequency λ. For instance, if data have a quarterly frequency, $\lambda = 4$ implies that each year we have one annual revision. If data are observed at the monthly frequency, one annual revision corresponds to $\lambda = 12$.

When the model is an AR(1), our percentile bootstrap provides adequately sized tests of the null regardless of annual revisions. In contrast, when the model is an AR(2), actual size of the test rises sharply, well above the nominal 5% level. As to power, in most cases the test rejects with frequencies near 95%. Even so, when the model is an AR(2) and there are no annual revisions, setting β to 0.5 does not constitute a deviation from the null and the rejection frequency aligns with the nominal size of the test.

7 Forecasting inflation

In this section we apply our bootstrap procedure to tests of equal forecast accuracy in the context of inflation forecasting. In particular, we do so with an eye toward revisiting Ang et al. (2007) who compare several forecasting methods and conclude that survey-based forecasts of inflation are generally superior. While their results do support that thesis, they consider only current vintage data and hence do not address a more realistic environment in which data are subject to revision.

With that in mind, we compare a small handful of linear forecasting models to survey forecasts of both CPI– and PCE–based quarterly inflation and do so using vintage data. The surveys consist of the Blue Chip (BC) and the Survey of Professional Forecasters (SPF). For the models we keep it simple and consider only two from Ang et al. (2007): an AR(2) and an AR(2) augmented with one lag of real GDP (RGDP) growth - the latter of which includes their preferred measure of economic slack. We also consider an AR(2) that includes one lag of the change in total capacity utilization (TCU), which is the preferred indicator of economic slack used by Stock and Watson (1999) who also conduct forecasting exercises using current, rather than real-time, vintages of data. Note that while revisions to CPI-based inflation are typically small, revisions to PCE-based inflation as well as RGDP and TCU can be substantial. Therefore, it is not immediately obvious that our results will align with those of Ang et al. (2007).

Vintages of the CPI and PCE price indices, RGDP growth, and TCU are obtained from the ALFRED database hosted by the Federal Reserve Bank of St. Louis. In each instance, the vintages are available monthly. We use the January, April, July, and October vintages exclusively as these are the first months for which (previous) quarter values can be constructed for all of the series. The SPF is obtained from the Federal Reserve Bank of Philadelphia and is released within the first two weeks of February, May, August, and November. The BC forecasts are obtained from the Haver database. While they are updated monthly, we use those vintages that align with the SPF. Together this implies a modest timing advantage for the surveys as they are released later than the implied forecast origins for the estimated models.[9]

Within each vintage t we apply the following data transformations. Annualized quarterly inflation ($y_t^{(1)} = y_t$) is constructed as four times the log difference of the price index associated with the last calendar month of quarters t and $t-1$. Annual inflation is constructed in the obvious way as $y_t^{(4)} = \sum_{j=0}^{3} y_{t-j}/4$. RGDP growth is quarterly percent change. TCU is transformed to be the average of the monthly first differences across the quarter.

[9] There are a few instances in which this timing is irregular. If there is more than one vintage within a month we use the latest release. If there is no release within a month, but there is one early in the subsequent month, we use it.

For each forecast horizon $\tau = 1, 4$, the three OLS estimated forecasting models take the form

$$y_t^{(\tau)}(t) = \beta_0 + \beta_1 y_{t-\tau}(t) + \beta_2 y_{t-\tau-1}(t) + \beta_3 x_{t-\tau}(t) + u_t^{(\tau)}(t)$$

where x is either omitted or denotes $RGDP$ or TCU. The forecast origins $t = R, \ldots, T$ vary depending on the target variables. When forecasting CPI inflation, the forecast origins range from 1996:Q4 through 2019:Q3, but when forecasting PCE inflation, they range from 2007:Q4 through 2019:Q3.[10] The distinction arises because the SPF started forecasting PCE inflation in 2007:Q4, and for a given target variable, we wanted to maintain a common sample across the three models and two surveys. For CPI, we use the most recent 50 observations to estimate the model at the first origin to restrict attention to a Great Moderation sample. For PCE, we continue to use an initial sample size of 50 for continuity across applications. Throughout, the forecasts are evaluated against the initial release $y_{t+\tau|1}^{(\tau)}$.

Table 7 provides the results of our forecasting exercise. The first two columns denote the ten pairwise model (survey) comparisons while the remaining columns distinguish the target variable and horizon. For each permutation of model comparison, target variable, and horizon we report three numbers. The first denotes the ratio of root mean squared errors (rmse) such that a value less than one favors model (survey) 1. The second number denotes the percentile bootstrapped p-value (in parentheses) associated with the test of equal forecast accuracy under quadratic loss.[11] For the same test, the third number is the p-value (in brackets) implied by the asymptotic distribution of the test statistic delineated in Clark and McCracken (2009).

In nominal terms, of the forty potential pairwise comparisons, all but seven of the RMSE ratios are quite close to one. In all of these instances, the bootstrapped and asymptotic p-values coincide in the sense that there are no cases where one implies a rejection of the null of equal forecast accuracy at any standard level of significance, while the other does not. Even so, for these instances, the conclusion reached by Ang et al. (2007) remains valid – surveys tend to be more accurate than the models albeit only modestly.

The exceptions all arise when forecasting PCE-based inflation at the longest horizon. Here we find that the models actually perform better than both surveys, and by a substantial margin. In accordance with the sizable difference in accuracy, the bootstrapped and asymptotic p-values are typically smaller than elsewhere in the table. In fact, they are often less than 10 or even 5 percent,

[10] Note that, for example, a Q4 forecast origin is based on the January (February) vintage of data when using the model (survey) to form a forecast.

[11] For ease of comparison, we use the bootstrap algorithm for linear models regardless of whether the models are nested or non-nested. When a survey is used it is paired with the dependent variable to preserve dependence between the two. In all comparisons, the algorithm shares a common seed across all 999 bootstrap replications.

Table 7: Application to Forecasting Inflation

Model 1	Model 2	CPI		PCE	
		$\tau = 1$	$\tau = 4$	$\tau = 1$	$\tau = 4$
AR(2)	AR(2) + RGDP	0.993	1.006	0.988	1.024
		(0.811)	(0.567)	(0.641)	(0.951)
		[0.239]	[0.820]	[0.144]	[0.370]
AR(2)	AR(2) + TCU	0.981	0.987	0.952	0.983
		(0.475)	(0.363)	(0.941)	(0.941)
		[0.515]	[0.344]	[0.277]	[0.627]
AR(2) + RGDP	AR(2) + TCU	0.988	0.981	0.964	0.960
		(0.439)	(0.907)	(0.969)	(0.983)
		[0.635]	[0.346]	[0.424]	[0.227]
BC	AR(2)	0.943	0.970	1.004	1.524
		(0.441)	(0.797)	(0.975)	(0.043)
		[0.198]	[0.749]	[0.975]	[0.023]
BC	AR(2) + RGDP	0.936	0.976	0.992	1.560
		(0.453)	(0.711)	(0.979)	(0.049)
		[0.160]	[0.820]	[0.953]	[0.019]
BC	AR(2) + TCU	0.925	0.957	0.956	1.498
		(0.747)	(0.669)	(0.695)	(0.053)
		[0.127]	[0.662]	[0.625]	[0.026]
SPF	AR(2)	0.949	0.980	0.971	1.233
		(0.543)	(0.715)	(0.363)	(0.111)
		[0.226]	[0.825]	[0.392]	[0.027]
SPF	AR(2) + RGDP	0.942	0.985	0.959	1.262
		(0.531)	(0.621)	(0.337)	(0.165)
		[0.182]	[0.888]	[0.238]	[0.028]
SPF	AR(2) + TCU	0.931	0.967	0.924	1.212
		(0.857)	(0.591)	(0.567)	(0.171)
		[0.147]	[0.728]	[0.148]	[0.108]
BC	SPF	0.994	0.990	1.034	1.236
		(0.179)	(0.455)	(0.847)	(0.197)
		[0.104]	[0.466]	[0.768]	[0.191]

Notes: For each pairwise comparison, the table presents: the ratio of root mean squared errors, the p-value associated with a test of equal forecast accuracy based on our percentile bootstrap (in parentheses), and the p-value for the same test based on the asymptotic distribution associated with the test statistic delineated in Clark and McCracken (2009) (in square brackets). RMSE ratios less (greater) than one favor model 1 (2). Results are provided for an initial window size $R = 50$ and horizons $\tau = 1$ and $\tau = 4$, across 999 bootstrap replications. The forecast origins range from 1996:Q4 to 2019:Q3 for CPI, and 2007:Q4 to 2019:Q3 for PCE.

suggesting statistically significant benefits to forecasting with the models rather than the surveys. This sharply differs from the conclusion in Ang et al. (2007), though it bears emphasizing that the sample we use is distinct from theirs. In particular, our sample for PCE inflation includes a period in which the unemployment rate declined substantially from a high of 10 percent to a low of 3.5 percent. In both surveys, especially the BC, longer horizon forecasts consistently overpredicted inflation as the unemployment rate fell, suggesting an overemphasis on a Phillip's curve relationship that has been a topic of substantial debate since the Great Recession (e.g., Del Negro et al., 2020).

8 Conclusions

The main contribution of this paper is to propose a new bootstrap algorithm for out-of-sample predictability tests when the data are subject to a finite number of data revisions. Our bootstrap algorithm replicates the triangular structure of the different vintages by relying on an application of the moving blocks bootstrap. The novel feature of our method is that it not only preserves the time series dependence of the data within each vintage, but it also preserves the dependence across the different vintages. We provide a set of regularity conditions under which our bootstrap method is asymptotically valid. Simulations show that the proposed bootstrap tests have comparable size properties to the data revision-robust test statistic proposed by Clark and McCracken (2009). However, the bootstrap is easier to apply as it avoids estimating directly the asymptotic variance of the test. We conclude with an application to inflation forecasting in the presence of real-time vintage data. In accordance with the simulations, our empirical results suggest comparable p-values associated with tests of equal forecast accuracy when using either bootstrap-based or asymptotic inference.

A Appendix

As usual in the bootstrap literature, we use P^* to denote the bootstrap probability measure, conditional on the original sample (defined on a given probability space $(\Omega, \mathcal{F}, P)$). For any bootstrap statistic t_T^*, we write $t_T^* = o_p^*(1)$, or $t_T^* \to^{P^*} 0$, when for any $\delta > 0$, $P^*(|t_T^*| > \delta) = o_p(1)$. We write $t_T^* = O_p^*(1)$, when for all $\delta > 0$ there exists $M_\delta < \infty$ such that $\lim_{T\to\infty} P[P^*(|t_T^*| > M_\delta) > \delta] = 0$. By Markov's inequality, this follows if $E^* |t_T^*|^q = O_p(1)$ for some $q > 0$. Finally, we write $t_T^* \to^{d^*} D$, in probability, if conditional on a sample with probability that converges to one, t_T^* weakly converges to the distribution D under P^*, i.e. $E^*(f(t_T^*)) \to^P E(f(D))$ for all bounded and uniformly continuous functions f.

For simplicity, we treat β as a scalar and focus on the case of a single model, i.e., we let $k = 1$ in Assumption 2. Following West (1996), we write "$\sup_t$" to mean "$\sup_{R\leq t\leq T}$."

A.1 Auxiliary lemmas

Here, we provide several auxiliary lemmas, followed by their proofs.

Lemma A.1 *Under Assumptions 1-5,*

(a) $\sup_t \left|\hat{B}(t) - B(t)\right| = o_p(1)$.

(b) $P^{-1/2} \sum_{t=R}^{T} \left|\hat{H}(t) - H(t)\right| = o_p(1)$.

(c) *For any* $0 \leq a < 1/2$, $\sup_t P^a |\hat{H}(t) - H(t)| = o_p(1)$.

(d) *For any* $0 \leq a < 1/2$, $\sup_t |P^a(\hat{\beta}(t) - \beta_0)| = o_p(1)$.

Lemma A.1(d) is the analog of Lemma A.3(b) of West (1996) when the estimator of β_0 is $\hat{\beta}(t)$, the OLS estimator based on real-time vintage data.

Lemma A.2 *Under Assumptions 1-5 and assuming that* $l \to \infty$ *such that* $l/\min\{\sqrt{R}, \sqrt{P}\} \to 0$,

(a) For any $0 \leq a < 1/2$, $\sup_t P^a |H^*(t) - E^* H^*(t)| = o_p^*(1)$, where $H^*(t) \equiv t^{-1} \sum_{s=1+\tau}^{t} h_s^*$, with $h_s^* \equiv x_{s-\tau}^*(y_s^* - x_{s-\tau}^{*\prime}\beta_0)$.

(b) $\sup_t |B^*(t) - B| = o_p^*(1)$, where $B^*(t) \equiv (t^{-1} \sum_{s=1+\tau}^{t} x_{s-\tau}^* x_{s-\tau}^{*\prime})^{-1}$.

(c) For any $0 \leq a < 1/2$, $\sup_t \left|P^a\left(\hat{\beta}_t^* - \beta_0\right)\right| = o_p^*(1)$.

(d) $P^{-1/2} \sum_{t=R}^{T} (f_{t+\tau|r',\beta}^* - F) B H^*(t) = o_p^*(1)$.

(e) $P^{-1/2} \sum_{t=R}^{T} F(B^*(t) - B) H^*(t) = o_p^*(1)$.

(f) $P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta}-F)(B^*(t)-B)H^*(t)=o^*_p(1)$.

Parts (a) and (c) are the bootstrap analogs of Lemma A.3 (a) and (b) of West (1996). Parts (d) through (f) are the bootstrap analogs of Lemma A.4 of West (1996).

To prove Lemma A.2, we rely on the following result. It provides an asymptotic approximation to the MBB expectation of a given observation in the MBB sample.

Lemma A.3 *Let $\{Z^*_t : t=M,\ldots,M'\}$ denote a MBB resample of $\{Z_t : t=N,\ldots,N'\}$ with block size l such that $l\to\infty$ with $l/n\to 0$, where $n=N'-N+1$. If $E|Z_t|\le\Delta$, $t=N,\ldots,N'$, for some $\Delta<\infty$, then $E^*(Z^*_t)=\bar{Z}_n+O_p(l/n)$, uniformly in $t=M,\ldots,M'$, where $\bar{Z}_n\equiv n^{-1}\sum_{t=N}^{N'}Z_t$.*

Lemma A.4 *Under Assumptions 1-5 and $l\to\infty$ such that $l/\min\{\sqrt{R},\sqrt{P}\}\to 0$,*

(a) $Var^*(S^*_{1P})\xrightarrow{p}\Omega_1$.

(b) $Var^*(S^*_{2P})\xrightarrow{p}\Omega_2$.

(c) $Cov^*(S^*_{1P},S^*_{2P})\xrightarrow{p}\Omega_{12}$.

Proof of Lemma A.1. To prove (a), it suffices to show that $\sup_t\left|\hat{B}^{-1}(t)-B^{-1}(t)\right|=o_p(1)$. We can write

$$\hat{B}^{-1}(t)-B^{-1}(t)=t^{-1}\sum_{s=1+\tau}^{t}\left(x_{s-\tau}(t)x_{s-\tau}(t)'-x_{s-\tau}x'_{s-\tau}\right)\equiv t^{-1}V_t,$$

where

$$V_t\equiv\sum_{s=t-r+1}^{t}(x_{s-\tau}(t)x_{s-\tau}(t)'-x_{s-\tau}x'_{s-\tau}).$$

Hence, for any $\varepsilon>0$,

$$P\left(\sup_t|t^{-1}V_t|>\varepsilon\right)\le\sum_{t=R}^{T}P\left(|t^{-1}V_t|>\varepsilon\right)\le\varepsilon^{-2}\sum_{t=R}^{T}t^{-2}E|V_t|^2.$$

The result follows by noting that $\sum_{t=R}^{T}t^{-2}\le PR^{-2}\to 0$ and $E|V_t|^2=O(1)$ by Assumption 4. Part (b) follows similarly by writing $t^{-1}V_t\equiv\hat{H}(t)-H(t)=t^{-1}\sum_{s=t-r+1}^{t}(h_s(t)-h_s)$. Part (c) follows similarly. For part (d),

$$\sup_t|P^a(\hat{\beta}(t)-\beta_0)|\le\sup_t|P^a(\hat{\beta}(t)-\hat{\beta}_t)|+\sup_t|P^a(\hat{\beta}_t-\beta_0)|,$$

where $\sup_t|P^a(\hat{\beta}(t)-\hat{\beta}_t)|=o_p(1)$ given parts (a) and (c), and $\sup_t|P^a(\hat{\beta}_t-\beta_0)|=o_p(1)$ by West's (1996) Lemma A.3(b). ■

Proof of Lemma A.2. Part (a). First, write

$$H^*(t) = \frac{1}{t}\sum_{s=1+\tau}^{R} h_s^* + \mathbf{1}_{\{t\geq R+1\}}\frac{1}{t}\sum_{s=R+1}^{t} h_s^*$$

where $\mathbf{1}_{\{t\geq R+1\}}$ is an indicator function equal to 1 if $t \geq R+1$. This decomposition is useful because $h_s^* = h_{\gamma_s}$ for $s \leq R$, whereas $h_s^* = h_{\eta_s}$ for $s > R$. Using it, for any $0 \leq a < 1/2$, we write

$$P^a \sup_t |H^*(t) - E^*H^*(t)| \leq \mathcal{A}_1^* + \mathcal{A}_2^*,$$

where

$$\mathcal{A}_1^* = P^a \frac{1}{\sqrt{R}}\left|\frac{1}{\sqrt{R}}\sum_{s=1+\tau}^{R}(h_s^* - E^*h_s^*)\right|, \ \mathcal{A}_2^* = P^a \sup_{R+1\leq t\leq T}\left|\frac{1}{t}\sum_{s=R+1}^{t}(h_s^* - E^*h_s^*)\right|,$$

where $\mathcal{A}_1^* = o_p^*(1)$ since $\left|R^{-1/2}\sum_{s=1+\tau}^{R}(h_s^* - E^*h_s^*)\right| = O_p^*(1)$. This follows by Chebyshev's inequality, since under Assumption 3 we can show that $Var^*\left(R^{-1/2}\sum_{s=1+\tau}^{R} h_s^*\right) = O_p(1)$ by Corollary 3.1 of Fitzenberger (1998). We are left to show that $\mathcal{A}_2^* = o_p^*(1)$. For simplicity, we assume that the number of blocks of size l needed to obtain the $t-R$ observations indexed by $s = R+1,\ldots,t$ is k (where $R+1 \leq t \leq T$), i.e. $t - R = kl$. Note that k is such that $1 \leq k \leq k_2$, since we have defined k_2 as the number of blocks of size l needed to obtain the last $T+\tau-(R+1)+1 = P+\tau-1$ bootstrap observations in the sample. With this notation, we can write

$$\mathcal{A}_2^* = P^a \sup_{1\leq k\leq k_2}\left|\frac{1}{kl+R}\sum_{s=R+1}^{R+kl}(h_s^* - E^*h_s^*)\right| \leq P^a R^{-1}\sup_{1\leq k\leq k_2}\left|\sum_{i=1}^{k}\mathcal{U}_i^*\right|,$$

where

$$\mathcal{U}_i^* \equiv \sum_{t=R+1+(i-1)l}^{R+(i-1)l+l}(h_t^* - E^*h_t^*) = \sum_{j=1}^{l}(h_{J_i+j-1} - E^*(h_{J_i+j-1})).$$

The last equality uses the fact that for $t = R+1+(i-1)\,l,\ldots,R+(i-1)l+l$,

$$h_t^* = h_{\eta_{R+1+(i-1)l+(j-1)}} = h_{J_i+j-1},$$

where $J_i \sim$ i.i.d. Uniform on $\{R+\tau,\ldots,T+\tau-l+1\}$. To prove that $\mathcal{A}_2^* = o_p^*(1)$, it suffices to show that $\sup_{1\leq k\leq k_2}|P^{-1/2}\sum_{i=1}^{k}\mathcal{U}_i^*| = O_p^*(1)$. To prove this, note that by the independence of $\{J_i\}$, $\{\mathcal{U}_i^* : i = 1,\ldots,k\}$ is an array of independent variables, implying that it is a martingale difference array with respect to the σ-field $\mathcal{G}_{i-1}^* = \sigma(J_1,\ldots,J_{i-1})$. Using Theorem 15.14 of Davidson (1994), for any $\epsilon > 0$,

$$P^*\left(\sup_{1\leq k\leq k_2}\left|P^{-1/2}\sum_{i=1}^{k}\mathcal{U}_i^*\right| > \epsilon\right) \leq \frac{E^*\left|P^{-1/2}\sum_{i=1}^{k_2}\mathcal{U}_i^*\right|^2}{\epsilon^2},$$

where

$$E^*\left(P^{-1/2}\sum_{i=1}^{k_2}\mathcal{U}_i^*\right)^2 = E^*\left(P^{-1/2}\sum_{s=R+1}^{T+\tau}(h_s^*-E^*h_s^*)\right)^2 = Var^*\left(P^{-1/2}\sum_{t=R+1}^{T+\tau}h_s^*\right),$$

which is $O_p(1)$ by Corollary 3.1 of Fitzenberger (1998). The result follows by noting that $P^aR^{-1}P^{1/2} = o\,(1)$ under Assumption 5 and $a < 1/2$.

Part (b). It suffices to show that $\sup_t |B^*(t)^{-1} - B^{-1}| = o_p^*(1)$, which follows if $\sup_t |B^*(t)^{-1} - E^*B^*(t)^{-1}| = o_p^*(1)$ and $\sup_t |E^*B^*(t)^{-1} - B^{-1}| = o_p^*(1)$. Let $a_s^* = x_{s-\tau}^*x_{s-\tau}^{*}{}'$, and $a_s = x_{s-\tau}x_{s-\tau}'$. By the triangle inequality,

$$\sup_t\left|B^*(t)^{-1} - E^*B^*(t)^{-1}\right| \le \left|R^{-1}\sum_{s=1+\tau}^{R}(a_s^*-E^*a_s^*)\right| + \sup_{R+1\le t\le T}\left|t^{-1}\sum_{s=R+1}^{t}(a_s^*-E^*a_s^*)\right|.$$

We can show that the two terms on the right-hand-side (RHS) of the inequality are $o_p^*(1)$ by relying on an argument similar to that used in the proof of part (a). Thus, we only need to show that $\sup_t |E^*B^*(t)^{-1} - B^{-1}| = o_p(1)$. Noting that $E^*B^*\left(t\right)^{-1} = t^{-1}\sum_{s=1+\tau}^{t}E^*a_s^*$ and $B^{-1} = Ea_s$ (which is constant under our stationarity assumption on x_t), we can write

$$\begin{aligned}\sup_t\left|E^*B^*(t)^{-1} - B^{-1}\right| &= \sup_t\left|t^{-1}\sum_{s=1+\tau}^{t}E^*(a_s^*-Ea_s) + \frac{t-\tau}{t}E\left(a_s\right) - E\left(a_s\right)\right| \\ &\le \sup_t\left|t^{-1}\sum_{s=1+\tau}^{t}E^*(a_s^*-Ea_s)\right| + O\left(R^{-1}\right),\end{aligned}$$

where the last inequality uses the fact that τ is finite. Noting that $a_s^* = a_{\gamma_s}$ for $s = 1+\tau,\ldots,R$ and $a_s^* = a_{\eta_s}$ for $s \ge R+1$, we can apply the triangular inequality to obtain

$$\sup_t\left|t^{-1}\sum_{s=1+\tau}^{t}E^*(a_s^*-Ea_s)\right| \le \left|\underbrace{R^{-1}\sum_{s=1+\tau}^{R}E^*(a_s^*-Ea_s)}_{\equiv\xi_1}\right| + \underbrace{\sup_t\left|t^{-1}\sum_{s=R+1}^{t}E^*(a_s^*-Ea_s)\right|}_{\equiv\xi_2}.$$

The first term is the absolute value of ξ_1, which can be rewritten as

$$\xi_1 \equiv \frac{R-\tau}{R}(R-\tau)^{-1}\sum_{s=1+\tau}^{R}E^*(a_s^*-Ea_s),$$

where $(R-\tau)^{-1}\sum_{s=1+\tau}^{R}E^*(a_s^*-Ea_s)$ is the MBB sample average of $a_s^*-Ea_s$. By well-known properties of the MBB (see e.g. Fitzenberger, 1998), we can show that this is equal to $(R-\tau)^{-1}\sum_{s=1+\tau}^{R}(a_s - Ea_s)+O_p\left(\frac{l}{R-\tau}\right)$. Hence, $\xi_1 = R^{-1}\sum_{s=1+\tau}^{R}(a_s-Ea_s)+O_p\left(l/R\right)$. We can show that $\xi_1 = O_p\left(R^{-1/2}\right)+o_p\left(1\right) = o_p\left(1\right)$ under Assumption 3 and the fact that $l/R = o\,(1)$. That $R^{-1}\sum_{s=1+\tau}^{R}(a_s - Ea_s)$ is $O_p\left(R^{-1/2}\right)$ follows by applying a maximal inequality for mixingales (see e.g. Lemma A.1 of Goncalves and Vogelsang (2011)). This result follows if $a_s - Ea_s \equiv x_{s-\tau}x'_{s-\tau} - Ex_{s-\tau}x'_{s-\tau}$ is an L_2-mixingale of size -1 (which is ensured by Assumption 3). We study ξ_2 next. This term relies on the MBB indices η_s. Using again the simplified assumption that $t - R = kl$, we can write

$$\sum_{s=R+1}^{t} E^*(a_s^* - Ea_s) = \sum_{i=1}^{k}\sum_{j=1}^{l} E^*\left(a_{J_i+(j-1)} - Ea_s\right) \text{ (where we note that } Ea_s \text{ is a constant)}$$

$$= l\sum_{i=1}^{k} E^*\left(\underbrace{l^{-1}\sum_{j=1}^{l}\left(a_{J_i+(j-1)} - Ea_s\right)}_{\equiv \mathcal{U}_i^*}\right) = klE^*(\mathcal{U}_1^*),$$

where the last equality holds because $\mathcal{U}_i^*$ is i.i.d. across i. Hence, we get that

$$\xi_2 \equiv \sup_{R+1\le t\le T}\left|t^{-1}\sum_{s=R+1}^{t} E^*(a_s^* - Ea_s)\right| \le R^{-1}\sup_{1\le k\le k_2}|klE^*(\mathcal{U}_1^*)| \le R^{-1}\underbrace{(k_2 l)}_{P-\tau+1}|E^*(\mathcal{U}_1^*)|.$$

The result follows by Assumption 5 (which implies that $R^{-1}(P-\tau+1) = O(1)$) and the fact that $E^*(\mathcal{U}_1^*) = O_p\left(P^{-1/2}\right)$. The latter follows because we can write

$$E^*(\mathcal{U}_1^*) = k_2^{-1}\sum_{i=1}^{k_2} E^*\left(\mathcal{U}_i^*\right) = E^*\left\{(k_2 l)^{-1}\sum_{i=1}^{k_2}\sum_{j=1}^{l}(a_{J_i+(j-1)} - Ea_s)\right\} = \frac{P}{P-\tau+1}E^*\left(P^{-1}\sum_{t=R+\tau}^{T+\tau}(a_t^* - Ea_t)\right),$$

which is the MBB bootstrap expectation of the sample average of $\{a_t^* - Ea_t : t = R+\tau, \ldots, T+\tau\}$ (note that $Ea_t = Ea_s$ since this is a constant under the stationarity assumption on x_t). Thus, by the properties of the MBB bootstrap expectation, we can write $E^*(\mathcal{U}_1^*)$ as

$$E^*(\mathcal{U}_1^*) = P^{-1}\sum_{t=R+\tau}^{T+\tau}(a_t - Ea_t) + O_p\left(l/P\right) = O_p\left(P^{-1/2}\right) + o_p\left(1\right) = o_p\left(1\right),$$

since the first term after the first equality is $O_p\left(P^{-1/2}\right)$, under Assumption 3 (as explained above). This concludes the proof that $\xi_2 = o_p\left(1\right)$ and the proof of part b).

Part (c). By definition, $\hat{\beta}_t^* = \left(t^{-1}\sum_{s=1+\tau}^{t} x_{s-\tau}^* x_{s-\tau}^{*\prime}\right)^{-1} t^{-1}\sum_{s=1+\tau}^{t} x_{s-\tau}^* y_s^*$. Letting $y_s^* = x_{s-\tau}^{*\prime}\beta_0 + (y_s^* - x_{s-\tau}^{*\prime}\beta_0)$ and recalling the definitions of $B^*\left(t\right)$ and $H^*\left(t\right)$ yields $\hat{\beta}_t^* - \beta_0 = B^*\left(t\right)H^*\left(t\right)$. Hence,

$$\begin{aligned} P^a\sup_t|\hat{\beta}_t^* - \beta_0| =& P^a\sup_t|B^*(t)H^*(t)| \\ \le& \sup_t\left|B^*(t) - B\right|P^a\sup_t\left|H^*(t) - E^*H^*(t)\right| + BP^a\sup_t|E^*H^*(t)| \\ &+ \sup_t\left|B^*(t) - B\right|P^a\sup_t\left|E^*H^*(t)\right| + BP^a\sup_t|H^*(t) - E^*H^*(t)|. \end{aligned}$$

Since $\sup_t\left|B^*(t) - B\right| = o_p^*(1)$ by part (b), and for $0 \le a < 1/2$, $P^a\sup_t\left|H^*(t) - E^*H^*(t)\right| = o_p^*(1)$ by part (a), the result follows by showing that $P^a\sup_t|E^*H^*(t)| = o_p(1)$. This result follows by the exact same arguments as in the proof of part (b). In particular, we can decompose $\sup_t |E^*H^*(t)| \leq \chi_1 + \chi_2$ where $\chi_1 = O_p\left(R^{-1/2}\right) + O_p\left(l/R\right)$ and $\chi_2 = O_p\left(P^{-1/2}\right) + O_p\left(l/P\right)$ (the proof of these results is the same as that used to study ξ_1 and ξ_2 in part b)). It follows that

$$P^a \sup_t |E^*H^*(t)| = O_p(P^a R^{-1/2}) + O_p\left(P^a l/R\right) + O_p(P^a P^{-1/2}) + O_p\left(P^a l/P\right) = o_p\left(1\right),$$

since P and R are of the same order magnitude (by Assumption 5), and $a < 1/2$. This implies that the first and third terms are $o_p\left(1\right)$. The second and fourth terms are also $o_p\left(1\right)$ under our assumptions. For instance, for the last term, because $l \to \infty$ such that $l/\sqrt{P} = o\left(1\right)$, we can write $O_p\left(P^a l/P\right) = O_p(P^a P^{-1/2} l P^{-1/2}) = o_p\left(1\right)$ since $l/P^{1/2} = o\left(1\right)$ and $P^{a-1/2} = o\left(1\right)$.

Part (d). Adding and subtracting appropriately, we can center $f^*_{t+\tau|r'}$ and $H^*(t)$ around their bootstrap means, i.e., $P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta} - F)BH^*(t) = \sum_{i=1}^{4}\mathcal{F}_i$ where

$$\mathcal{F}_1 = P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta} - E^*f^*_{t+\tau|r',\beta})B(H^*(t) - E^*H^*(t)), \quad \mathcal{F}_2 = P^{-1/2}\sum_{t=R}^{T}(E^*f^*_{t+\tau|r',\beta} - F)BE^*H^*(t)$$

$$\mathcal{F}_3 = P^{-1/2}\sum_{t=R}^{T}(E^*f^*_{t+\tau|r',\beta} - F)B(H^*(t) - E^*H^*(t)), \text{ and } \mathcal{F}_4 = P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta} - E^*f^*_{t+\tau|r',\beta})BE^*H^*(t).$$

To prove part (d), it suffices to show that each $\mathcal{F}_i$ vanishes asymptotically. We start by showing $\mathcal{F}_2 = o_p(1)$. Note that $\mathcal{F}_2$ is bounded by

$$\mathcal{F}_2 \leq P^{-1/2}\sum_{t=R}^{T}\left|E^*f_{t+\tau|r',\beta} - F\right|B\sup_t\left|E^*H^*(t)\right|,$$

implying that $\mathcal{F}_2 = o_p(1)$ if $P^{-1/2}\sum_{t=R}^{T}\left|E^*f_{t+\tau|r',\beta} - F\right| = O_p(1)$ and $\sup_t\left|E^*H^*(t)\right| = o_p(1)$, where $\sup_t\left|E^*H^*(t)\right| = o_p(1)$ follows from the result of part (c). To show $P^{-1/2}\sum_{t=R}^{T}\left|E^*f_{t+\tau|r',\beta} - F\right| = O_p(1)$, it suffices to show that $P^{-1}\sum_{t=R}^{T}P^{1/2}E\left|E^*f_{t+\tau|r',\beta} - F\right| = O(1)$. This condition requires $P^{1/2}E\left|E^*f_{t+\tau|r',\beta} - F\right| = O(1)$ to hold for $t = R, \ldots, T$. Using Jensen's inequality, we can write

$$P^{1/2}E\left|E^*f^*_{t+\tau|r',\beta} - F\right| \leq P^{1/2}\left(E\left(E^*f^*_{t+\tau|r',\beta} - F\right)^2\right)^{1/2},$$

where for $t = R, \ldots, T$, $E\left(E^*f^*_{t+\tau|r',\beta} - F\right)^2 \leq O(1/P)$. Hence, $P^{1/2}E\left|E^*f^*_{t+\tau|r',\beta} - F\right| \leq O(1)$, completing the proof that $\mathcal{F}_2 = o^*_p(1)$.

For $\mathcal{F}_3$, we write

$$\mathcal{F}_3 \leq P^{-1}\sum_{t=R}^{T}P^{1/2}\left|E^*f^*_{t+\tau|r',\beta} - F\right|B\sup_t\left|H^*(t) - E^*H^*(t)\right|,$$

where $\sup_t\left|H^*(t) - E^*H^*(t)\right| = o^*_p(1)$ by Lemma A.2 (a) and $P^{1/2}\left|E^*f_{t+\tau|r',\beta} - F\right| = O_p(1)$. Hence, $\mathcal{F}_3 = o^*_p(1)$.

For $\mathcal{F}_4$, it suffices to show $E^*\mathcal{F}_4 = 0$ and $Var^*(\mathcal{F}_4) = o_p(1)$. Note that $E^*\mathcal{F}_4 = 0$ by design. Hence, we only need to show that $Var^*(\mathcal{F}_4) = o_p(1)$. For the sake of brevity, we let $\tau = 1$, and we define $a^*_{t+\tau} \equiv f^*_{t+\tau|r',\beta} - E^* f^*_{t+\tau|r',\beta}$, $c_t \equiv BE^*H^*(t)$. For $i = 1, \ldots, k_2$, we let $n_i = R + 1 + (i-1)l$, so $n_1 = R+1$ and $n_{k_2} = T+1-l+1$, where $R+1$ is the index that links to the first element in the first generated block, and $T+1-l+1$ is the index that links to the first element in the last generated block. Exploiting the independence between bootstrap blocks, we can write

$$Var^*(\mathcal{F}_4) = E^*\left(P^{-1/2}\sum_{t=R}^{T} a^*_{t+1}c_t\right)^2 = k_2^{-1}\sum_{i=1}^{k_2} l^{-1}E^*\left(\sum_{j=1}^{l} a^*_{n_i+(j-1)}c_{n_i+(j-1)-1}\right)^2.$$

Now, we let $d^*_{n_i+(j-1)} = a^*_{n_i+(j-1)}c_{n_i+(j-1)-1}$ and apply the standard MBB variance formula to get

$$Var^*(\mathcal{F}_4) = k^{-1}\sum_{i=1}^{k_2}\left(l^{-1}\sum_{j=1}^{l} E^* d^{*2}_{n_i+(j-1)} + 2l^{-1}\sum_{m=1}^{l-1}\sum_{j=1}^{l-m} E^*(d^*_{n_i+(j-1)}d^*_{n_i+(j-1+m)})\right),$$

where $l^{-1}\sum_{j=1}^{l} E^* d^{*2}_{n_i+(j-1)} \leq O_p(P^{-1/2})$ and $\sum_{j=1}^{l-m} E^*(d^*_{n_i+(j-1)}d^*_{n_i+(j-1+m)}) \leq O_p(l/P^{1/2})$ for $i = 1, \ldots, k_2$. For brevity, we only show $E^* d^{*2}_{n_i+(j-1)} = O_p(P^{-1/2})$ for $i = 1, \ldots, k_2$ and $j = 1, \ldots, l$. The proof that $\sum_{j=1}^{l-m} E^*(d^*_{n_i+(j-1)}d^*_{n_i+(j-1+m)}) = o_p(1)$ follows from a similar argument. Using the definition $d^*_{n_i+(j-1)} = a^*_{n_i+(j-1)}c_{n_i+(j-1)-1}$, we can write

$$\begin{aligned}
E^* d^{*2}_{n_i+(j-1)} &= Var^*(a^*_{n_i+(j-1)})c^2_{n_i+(j-1)-1} \\
&\leq \Big|Var^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\Big|c^2_{n_i+(j-1)-1} + |\Gamma_{aa}(0)|c^2_{n_i+(j-1)-1} \\
&\leq \Big|Var^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\Big|\left(B\sup_t|E^*H^*(t)|\right)^2 + |\Gamma_{aa}(0)|\left(B\sup_t|E^*H^*(t)|\right)^2
\end{aligned}$$

where $\Gamma_{aa}(0) \equiv Var(f_{t|r',\beta})$ and B are constants, $\sup_t |E^*H^*(t)| \leq O_p(P^{-1/2})$ by result of part (c). We are left to show that $E\Big|Var^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\Big| = O(1)$. Using Jensen's inequality,

$$\begin{aligned}
E\Big|Var^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\Big| &\leq \left[E\left(Var^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\right)^2\right]^{1/2} \\
&\leq \left[Var\left(Var^*(a^*_{n_i+(j-1)})\right) + \left(EVar^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0)\right)^2\right]^{1/2}
\end{aligned}$$

where for $i = 1, \ldots, k_2$,

$$Var^*(a^*_{n_i+(j-1)}) = \frac{1}{P+1-l}\sum_{t=R+1}^{T+1-l+1}\left(f_{t+(j-1)|r',\beta} - \mathcal{C}_{f,j-1}\right)^2 \text{ with } \mathcal{C}_{f,j-1} = \frac{1}{P+1-l}\sum_{s=R+1}^{T+1-l+1} f_{s+(j-1)|r',\beta}.$$

Using the uniform fourth moment bound on a_t, $Var\left(Var^*(a^*_{n_i+(j-1)})\right) \to 0$. Note that

$$EVar^*(a^*_{n_i+(j-1)}) - \Gamma_{aa}(0) = -E(\mathcal{C}_{f,j-1})^2 \leq O(1)$$

which completes this part of the proof.

For $\mathcal{F}_1$, it suffices to show $E^*(\mathcal{F}_1^2) = o_p(1)$. For simplicity, we let $\tau = 1$, $k_2 = 2$, and for $i = 1, 2$, $n_i = R + 1 + (i-1)l$. This simple setting implies that $(T+1) - (R+1) + 1 = 2l$. For $t = R, \ldots, T$, we let $a^*_{t+1} = f^*_{t+1|r',\beta} - E^* f^*_{t+1|r',\beta}$ and $c^*_t = B(H^*(t) - E^* H^*(t))$. Now we write

$$E^*(\mathcal{F}_1^2) = E^*\left(P^{-1/2}\sum_{t=R+1}^{T+1} a^*_t c^*_{t-1}\right)^2 = 2^{-1} E^*\left(\underbrace{l^{-1/2}\sum_{j=1}^{l} a^*_{n_1+(j-1)} c^*_{n_1+(j-2)}}_{A^*_1} + \underbrace{l^{-1/2}\sum_{j=1}^{l} a^*_{n_2+(j-1)} c^*_{n_2+(j-2)}}_{A^*_2}\right)^2$$

where $E^*(A^*_1 + A^*_2)^2 = E^*(A^*_1 A^*_1) + E^*(A^*_2 A^*_2) + 2E^*(A^*_1 A^*_2)$. The result follows by showing that $E(A^*_1 A^*_1)$, $E(A^*_2 A^*_2)$ and $E(A^*_1 A^*_2)$ all vanish asymptotically. Using the definition of c^*_t and $H^*(t)$, we can decompose A^*_1 and A^*_2 into $A^*_1 = A^*_{1.0} + A^*_{1.1}$ and $A^*_2 = A^*_{2.0} + A^*_{2.1} + A^*_{2.2}$, respectively, as

$$A^*_{1.0} = \underbrace{l^{-1/2}\sum_{j=1}^{l} a^*_{n_1+(j-1)} \frac{1}{n_1 + (j-2)}}_{\mathcal{M}^*_{1.0}} \underbrace{\left(\sum_{s=2}^{R} B(h^*_s - E^* h^*_s)\right)}_{\mathcal{N}^*_{1.0}}$$

$$A^*_{1.1} = l^{-1/2}\sum_{j=2}^{l} a^*_{n_1+(j-1)} \frac{1}{n_1 + (j-2)} \left(\sum_{s=n_1}^{n_1+(j-1)-1} B(h^*_s - E^* h^*_s)\right),$$

$$A^*_{2.0} = \underbrace{l^{-1/2}\sum_{j=1}^{l} a^*_{n_2+(j-1)} \frac{1}{n_2 + (j-2)}}_{\mathcal{M}^*_{2.0}} \underbrace{\left(\sum_{s=2}^{R} B(h^*_s - E^* h^*_s)\right)}_{\mathcal{N}^*_{1.0}=\mathcal{N}^*_{2.0}},$$

$$A^*_{2.1} = \underbrace{l^{-1/2}\sum_{j=1}^{l} a^*_{n_2+(j-1)} \frac{1}{n_2 + (j-2)}}_{\mathcal{M}^*_{2.0}=\mathcal{M}^*_{2.1}} \underbrace{\left(\sum_{s=n_1}^{n_1+l-1} B(h^*_s - E^* h^*_s)\right)}_{N^*_{2.1}}, \text{ and}$$

$$A^*_{2.2} = l^{-1/2}\sum_{j=2}^{l} a^*_{n_2+(j-1)} \frac{1}{n_2 + (j-2)} \left(\sum_{s=n_2}^{n_2+(j-1)-1} B(h^*_s - E^* h^*_s)\right).$$

These decompositions can be used to identify pairs of bootstrap terms that are independent. In particular, any two terms that do not share the same base index set are independent. For an example, $\mathcal{M}^*_{1,0}$ is independent of $\mathcal{N}^*_{1,0}$. Exploiting this independence, we can write

$$E^*(A^*_1 A^*_1) = E^*(A^*_{1.0} A^*_{1.0}) + E^*(A^*_{1.1} A^*_{1.1})$$
$$E^*(A^*_2 A^*_2) = E^*(A^*_{2.0} A^*_{2.0}) + E^*(A^*_{2.1} A^*_{2.1}) + E^*(A^*_{2.2} A^*_{2.2})$$
$$E^*(A^*_1 A^*_2) = E^*(A^*_{1.1} A^*_{2.2}).$$

To show $E^*(\mathcal{F}_1^2) = o^*_p(1)$, it suffices to show that all the terms on the RHS of the above equations vanish asymptotically. For instance, for $E^*(A^*_{1.0}A^*_{1.0})$, we write

$$E^*(A^*_{1.0}A^*_{1.0}) = E^*(R^{1/2}M^*_{1.0}R^{1/2}M^*_{1.0})E^*(R^{-1/2}\mathcal{N}^*_{1.0}R^{-1/2}\mathcal{N}^*_{1.0})$$

where $E^*(R^{-1/2}\mathcal{N}^*_{1.0}R^{-1/2}\mathcal{N}^*_{1.0}) = O_p(1)$ by Fitzenberger (1998) Corollary 3.1 and $E^*(R^{1/2}M^*_{1.0}R^{1/2}M^*_{1.0}) = o_p(1)$ since $R^{1/2}(n_1 + (j-2))^{-1} \leq R^{-1/2} < l^{-1/2}$ for $j = 1, \ldots, l$. Using a similar method, one can easily show $E^*(A^*_{2.0}A^*_{2.0}) = o_p(1)$ and $E^*(A^*_{2.1}A^*_{2.1}) = o_p(1)$. For $E^*(A^*_{1.1}A^*_{2.2})$, we use the block's independence and write

$$E^*(A^*_{1.1}A^*_{2.2}) = E^*(A^*_{1.1})E^*(A^*_{2.2})$$

where $E^*(A^*_{1.1}) = o_p(1)$ and $E^*(A^*_{2.2}) = o_p(1)$. This is because

$$E^*(A^*_{1.1}) \leq l^{-1}\sum_{j=2}^{l}\frac{l^{1/2}}{n_1+(j-2)}\sum_{s=n_1}^{n_1+(j-2)}\left|E^*\left(a^*_{n_1+(j-1)}B(h^*_s - E^*h^*_s)\right)\right|$$

where $\frac{l^{1/2}}{n_1+(j-2)} \leq \frac{l^{1/2}}{R}$ for $j = 2, \ldots, l$, and

$$\left|E^*\left(a^*_{n_1+(j-1)}B(h^*_s - E^*h^*_s)\right)\right| = \left|Cov^*(f^*_{n_1+(j-1)|r',\beta}, Bh^*_s)\right| \leq O_p(1)$$

for $s = n_1, \ldots, n_1 + l - 1$. Hence $E^*(A^*_{1.1}) \leq O_p(\frac{l}{R^{1/2}}\frac{l^{1/2}}{R^{1/2}})$ where $\frac{l}{R^{1/2}} \to 0$ and $\frac{l^{1/2}}{R^{1/2}} \to 0$. Using a similar logic, we can show that $E^*(A^*_{2.2}) = o_p(1)$. For $E^*(A^*_{1.1}A^*_{1.1})$, we directly compute the bootstrap moment. We let $\mathcal{D}^*_{n_1+(j-1)} = a^*_{n_1+(j-1)}\frac{1}{n_1+(j-2)}\sum_{s=n_1}^{n_1+(j-1)-1}B(h^*_s - E^*h^*_s)$ then

$$\begin{aligned}E^*(A^*_{1.1}A^*_{1.1}) &= l^{-1}\sum_{j=2}^{l}E^*\mathcal{D}^{*2}_{n_1+(j-1)} + 2l^{-1}\sum_{m=1}^{l-2}\sum_{j=2}^{l-m}E^*(\mathcal{D}^*_{n_1+(j-1)}\mathcal{D}^*_{n_1+(j-1+m)}) \\ &\leq l^{-1}\sum_{j=2}^{l}\left|E^*\mathcal{D}^{*2}_{n_1+(j-1)}\right| + 2l^{-1}\sum_{m=1}^{l-2}\sum_{j=2}^{l-m}\left|E^*(\mathcal{D}^*_{n_1+(j-1)}\mathcal{D}^*_{n_1+(j-1+m)})\right|,\end{aligned}$$

where $\left|E^*\mathcal{D}^{*2}_{n_1+(j-1)}\right| = o_p(1)$ and $\sum_{j=2}^{l-m}\left|E^*(\mathcal{D}^*_{n_1+(j-1)}\mathcal{D}^*_{n_1+(j-1+m)})\right| = o_p(1)$. Note that for $j = 2, \ldots, l$

$$\begin{aligned}\left|E^*\mathcal{D}^{*2}_{n_1+(j-1)}\right| &= \left|E^*\left(a^*_{n_1+(j-1)}\frac{1}{n_1+(j-2)}\sum_{s=n_1}^{n_1+(j-1)-1}B(h^*_s - E^*h^*_s)\right)^2\right| \\ &\leq R^{-2}\left|E^*\left(\sum_{s=n_1}^{n_1+(j-1)-1}a^*_{n_1+(j-1)}B(h_s - E^*h^*_s)\right)^2\right| \leq O_p\left(\frac{l^2}{R^2}\right).\end{aligned}$$

We can also show that $\sum_{j=2}^{l-m}\left|E^*(\mathcal{D}^*_{n_1+(j-1)}\mathcal{D}^*_{n_1+(j-1+m)})\right| \leq O_p(l^3/R^2)$ where $l^3/R^2 \to 0$ under our assumption on l. In particular, here we provide the exact form of $E^*\mathcal{D}^{*2}_{n_1+(j-1)}$ for $j = 2$. When $j = 2$, $E^*\mathcal{D}^2_{n_1+(2-1)} = \frac{1}{n_1^2}E^*\left(a^*_{n_1+1}B(h^*_{n_1} - E^*h^*_{n_1})\right)^2$, where

$$E^*\left(a^*_{n_1+1}B(h^*_{n_1} - E^*h^*_{n_1})\right)^2 = \frac{1}{P-l+1}\sum_{t=R+1}^{T+1-l+1}\left(f_{t+1|r',\beta} - \mathcal{C}_{f,2}\right)^2\left(Bh_t - B\mathcal{C}_{h,1}\right)^2$$

where $\mathcal{C}_{f,2} = \frac{1}{P-l+1}\sum_{s=R+1}^{T+1-l+1} f_{s+1|r',\beta}$, and $\mathcal{C}_{h,1} = \frac{1}{P-l+1}\sum_{s=R+1}^{T+1-l+1} h_s$.

Part (e). Note that

$$\begin{aligned}P^{-1/2}\sum_{t=R}^{T} F(B^*(t)-B)H^*(t) \leq &\sup_t |B^*(t)-B|FP^{-1/2}\sum_{t=R}^{T}|H^*(t)-E^*H^*(t)| \\ &+\sup_t |B^*(t)-B|FP^{-1/2}\sum_{t=R}^{T}|E^*H^*(t)|,\end{aligned}$$

where $\sup_t |B(t)^*-B| = o_p^*(1)$ by part (b). The result follows by showing that (i) $P^{-1/2}\sum_{t=R}^{T}|H^*(t)-E^*H^*(t)| = O_p^*(1)$ and (ii) $P^{-1/2}\sum_{t=R}^{T}|E^*H^*(t)| = O_p(1)$. To prove (i), it suffices to show $E^*|H^*(t)-E^*H^*(t)| \leq O_p(R^{-1/2})$ uniformly in t. For $t = R,\ldots,T$, observe that $E^*|H^*(t)-E^*H^*(t)| \leq \left[E^*\big(H^*(t)-E^*H^*(t)\big)^2\right]^{1/2}$, where

$$\begin{aligned}E^*\big(H^*(t)-E^*H^*(t)\big)^2 =&E^*\left(t^{-1}\sum_{s=1+\tau}^{R}(h_s^*-E^*h_s^*)+\frac{\mathbf{1}_{\{R+1\leq t\leq T\}}}{t}\sum_{s=R+1}^{t}(h_s^*-E^*h_s^*)\right)^2 \\ =&\frac{R}{t^2}E^*\left(R^{-1/2}\sum_{s=1+\tau}^{R}(h_s^*-E^*h_s^*)\right)^2 \\ &+\frac{\mathbf{1}_{\{R+1\leq t\leq T\}}(t-R)}{t^2}E^*\left((t-R)^{-1/2}\sum_{s=R+1}^{t}(h_s^*-E^*h_s^*)\right)^2\end{aligned}$$

where $\mathbf{1}_{\{R+1\leq t\leq T\}}$ is an indicator function that equals 1 if $R+1\leq t\leq T$. Note that $\frac{R}{t^2}\leq R^{-1}$, $(t-R)t^{-2}\leq PR^{-2} = O(R^{-1})$, $Var^*(R^{-1/2}\sum_{s=1+\tau}^{R}h_s^*) = O_p(1)$, $Var^*((t-R)^{-1/2}\sum_{s=R+1}^{t}h_s^*) = O_p(1)$, and $Cov^*(R^{-1/2}\sum_{s=1+\tau}^{R}h_s^*,(t-R)^{-1/2}\sum_{s=R+1}^{t}h_s^*) = 0$. To complete the proof, we show (ii). This follows from noting that

$$P^{-1/2}\sum_{t=R}^{T}|E^*H^*(t)| \leq P^{1/2}\sup_t |E^*H^*(t)| = O_p\left(1\right),$$

since we already showed that $\sup_t |E^*H^*(t)| \leq \chi_1+\chi_2 = O_p\left(R^{-1/2}\right)+O_p\left(l/R\right)+O_p(P^{-1/2})+O_p\left(l/P\right)$ in part (c).

Part (f). Adding and subtracting appropriately,

$$\begin{aligned}P^{-1/2}\sum_{t=R}^{T}(f_{t+\tau|r',\beta}^*-F)\Big(B^*(t)-B\Big)H^*(t) \leq &\sup_t \Big|B^*(t)-B\Big|P^{-1/2}\sum_{t=R}^{T}\Big|f_{t+\tau|r',\beta}^*-F\Big|\,|H^*(t)-E^*H^*(t)| \\ &+\sup_t \Big|B^*(t)-B\Big|P^{-1/2}\sum_{t=R}^{T}\Big|f_{t+\tau|r',\beta}^*-F\Big|\,|E^*H^*(t)|.\end{aligned}$$

Given part b), it suffices to show (i) $P^{-1/2}\sum_{t=R}^{T}\Big|f_{t+\tau|r',\beta}^*-F\Big||H^*(t)-E^*H^*(t)| = O_p^*(1)$ and (ii) $P^{-1/2}\sum_{t=R}^{T}\Big|f_{t+\tau|r',\beta}^*-F\Big|\,|E^*H^*(t)| = O_p^*(1)$. We can prove (i) by applying the Cauchy-Schwarz inequality and using the fact that $E^*\left(H^*(t) - E^*H^*(t)\right)^2 = O_p\left(\min(R,P)^{-1/2}\right) + O_p\left(l/\min(R,P)\right)$, as in the proof of part (e). Part (ii) follows by noting that $E^*\left(f^*_{t+\tau|r',\beta} - F\right)^2 = O_p(1)$ (by Lemma A.3) and the fact that $\sup_t |E^*H^*(t)| = O_p\left(\min(R,P)^{-1/2}\right) + O_p\left(l/\min(R,P)\right)$. ■

Proof of Lemma A.3. Let $M' - M + 1 = kl$ and generate $I_1, \ldots, I_k \sim$ i.i.d. Uniform on $\{N, \ldots, N' - l + 1\}$. Set

$$Z^*_{M+(i-1)l+(j-1)} = Z_{I_i+(j-1)}, \text{ for } i = 1, \ldots, k,\ j = 1, \ldots, l,$$

and note that this yields a bootstrap sample

$$\left\{Z^*_M = Z_{I_1}, Z^*_{M+1} = Z_{I_1+1}, \ldots, Z^*_{M+l-1} = Z_{I_1+l-1}, Z^*_{M+l} = Z_{I_2}, \ldots, Z^*_{M'} = Z_{I_k+l-1}\right\}.$$

Since I_i is i.i.d. Uniform on $\{N, \ldots, N' - l + 1\}$, it follows that for $i = 1, \ldots, k$, $j = 1, \ldots, l$,

$$E^*(Z^*_{M+(i-1)l+(j-1)}) = E^*(Z_{I_i+(j-1)}) = (N' - N - l + 2)^{-1} \sum_{t=N}^{N'-l+1} Z_{t+(j-1)},$$

which varies with $j = 1, \ldots, l$, but not with $i = 1, \ldots, k$. Let $n = N' - N + 1$ and $\bar{Z}_n \equiv n^{-1} \sum_{t=N}^{N'} Z_t$. For each $j = 1, \ldots, l$, we can write

$$(N' - N - l + 2)^{-1} \sum_{t=N}^{N'-l+1} Z_{t+(j-1)} = \frac{n}{n - l + 1} \left(n^{-1} \sum_{t=N}^{N'} Z_t\right) + (n - l + 1)^{-1} A_j,$$

where $A_j \equiv \sum_{t=N}^{N'-l+1} Z_{t+(j-1)} - \sum_{t=N}^{N'} Z_t$. Each term A_j can be written as a sum involving $l - 1$ observations in $\{Z_t\}$. For instance, for $j = 1$,

$$(n - l + 1)^{-1} A_1 = \frac{l-1}{n-l+1} \underbrace{(l-1)^{-1}(Z_{N'-l+2} + \ldots + Z_{N'})}_{a_1 = O_p(1)} = O_p(l/n) \text{ if } l/n = o(1),$$

where $a_1 = O_p(1)$ because $E|Z_t| \leq \Delta$ for all t. For $j = 2$,

$$(n - l + 1)^{-1} A_2 = \frac{l-1}{n-l+1} \underbrace{(l-1)^{-1}(Z_{N+1} + Z_{N'-l+3} + \ldots + Z_{N'})}_{a_2 = O_p(1)} = O_p(l/n) \text{ if } l/n = o(1).$$

Since, for any j, we can show that $(n - l + 1)^{-1} E|A_j| = O(l/n)$ uniformly in j, the result follows. ■

Proof of Lemma A.4. **Part (a).** Recall that $S^*_{1P} = P^{-1/2} \sum_{t=R+\tau}^{T+\tau} (f^*_{t|r'} - f_{t|r'})$. We start by showing $S^*_{1P} = \tilde{S}^*_{1P} + o^*_p(1)$ where $\tilde{S}^*_{1P} \equiv P^{-1/2} \sum_{t=R+1}^{T+\tau} (f^*_{t|r'} - E^* f^*_{t|r'})$. Using $\tilde{S}^*_{1P}$ helps to set the first summand of S^*_{1P} from $f^*_{R+\tau|r'}$ to $f^*_{R+1|r'}$ where $f^*_{R+1|r'} = f_{\eta_{R+1}|r'}$ is the first element of the first random block based on $\{\eta_{R+1}, \ldots, \eta_{R+1} + (l-1)\}$, and $\tilde{S}^*_{1P}$ is centered around bootstrap mean. Adding and subtracting appropriately,

$$S^*_{1P} = \tilde{S}^*_{1P} + P^{-1/2}\Delta^* + P^{-1/2}\Delta$$

where $\Delta^* = \sum_{t=R+\tau}^{T+\tau} f^*_{t|r'} - \sum_{s=R+1}^{T+\tau} f^*_{s|r'}$, $\Delta = \sum_{s=R+1}^{T+\tau} Ef^*_{s|r'} - \sum_{t=R+\tau}^{T+\tau} f_{t|r'}$. Note that Δ^* is at most $O_p^*(\tau)$, and it is exactly zero when $\tau = 1$. Hence $P^{-1/2}\Delta^*$ vanishes asymptotically. Let $\mathcal{C}_f = P^{-1}\sum_{t=R+\tau}^{T+\tau} f_{t|r'}$ then

$$\Delta = \sum_{s=R+1}^{T+\tau} \left(E^* f_{s|r'} - \mathcal{C}_f + \mathcal{C}_f\right) - P\mathcal{C}_f = \sum_{s=R+1}^{T+\tau} \left(E^* f_{s|r'} - \mathcal{C}_f\right) + (\tau - 1)\mathcal{C}_f$$

where $E^* f_{s|r'} - \mathcal{C}_f \leq O_p(l/P)$ for $t = R+1, \ldots, T+\tau$ by Lemma A.3, and $\mathcal{C}_f \leq O_p(1)$. This implies $P^{-1/2}\Delta \leq P^{-1/2}(P+\tau-1)O_p(l/P) + P^{-1/2}(\tau-1)O_p(1)$. Hence, $P^{-1/2}\Delta$ vanishes asymptotically under the block length condition $l/\sqrt{P} \to 0$. Using these results, we can write $S^*_{1P} = \tilde{S}^*_{1P} + o_p^*(1)$. This implies

$$\lim_{R,P\to\infty} Var^*\left(S^*_{1P}\right) = \lim_{R,P\to\infty} Var^*\left(\tilde{S}^*_{1P}\right)$$

where

$$Var^*\left(\tilde{S}^*_{1P}\right) \xrightarrow{p} \lim_{R,P\to\infty} Var\left(P^{-1/2}\sum_{t=R+\tau}^{T+\tau} f_{t|r'}\right) = \Omega_1$$

by Corollary 3.1 of Fitzenberger (1998).

Part (b). From Lemma 5.1, we know that

$$S^*_{2P} = \underbrace{a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h^*_s - \bar{h}_R)}_{S^*_{2P.1}} + \underbrace{P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(h^*_{R+i} - \bar{h}_P)}_{S^*_{2P.2}}.$$

We first recenter S^*_{2P} by adding and subtracting the appropriate bootstrap mean of $S^*_{2P.1}$ and $S^*_{2P.2}$

$$S^*_{2P.1} = a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h^*_s - E^*h^*_s) + a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(E^*h^*_s - \bar{h}_R)$$

$$S^*_{2P.2} = P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(h^*_{R+i} - E^*h^*_{R+i}) + P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(E^*h^*_{R+i} + \bar{h}_P)$$

where $a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(E^*h^*_s - \bar{h}_R) = o_p(1)$ since $a_{R,0} < \infty$ and $E^*h^*_s - \bar{h}_R = O_p(l/R)$ for $s = 1+\tau, \ldots, R$, and $P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(E^*h^*_{R+i} - \bar{h}_P) = o_p(1)$ since $P^{-1}\sum_{i=1}^{P-1} a_{R,i} \to 1 - \pi^{-1}\ln(1+\pi)$ (see West (1996), Lemma 4.1) and $E^*h^*_{R+i} - \bar{h}_P = O_p(l/R)$ for $i = 1, \ldots, P-1$. Now, we can write

$$S^*_{2P} = \underbrace{a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h^*_s - E^*h^*_s)}_{\tilde{S}^*_{2P.1}} + \underbrace{P^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(h^*_{R+i} - E^*h^*_{R+i})}_{\tilde{S}^*_{2P.2}} + o_p^*(1),$$

and

$$\lim_{R,P\to\infty} Var^*\left(S^*_{2P}\right) = \lim_{R,P\to\infty}\left(Var^*\left(\tilde{S}^*_{2P.1}\right) + Var^*\left(\tilde{S}^*_{2P.2}\right) + 2Cov^*\left(\tilde{S}^*_{2P.1}, \tilde{S}^*_{2P.2}\right)\right).$$

Note that

$$\lim_{R,P\to\infty} Var^*(\tilde{S}^*_{2P.1}) = \lim_{R,P\to\infty} a^2_{R,0} P^{-1} R Var^* \left(R^{-1/2} \sum_{s=1+\tau}^{R} (h^*_s - E^* h^*_s) \right)$$

where $a^2_{R,0} P^{-1} R \to \pi^{-1} \ln^2(1+\pi)$ by West (1996), page 1082 (A-1a); by using Fitzenberger's (1998) Corollary 3.1,

$$Var^* \left(R^{-1/2} \sum_{t=1+\tau}^{R} (h^*_t - E^* h^*_s) \right) \xrightarrow{p} Var \left(R^{-1/2} \sum_{t=1+\tau}^{R} h_t \right) \to \sum_{j=-\infty}^{\infty} \Gamma_{hh}(j)$$

where $\Gamma_{hh}(j) \equiv E(h_t h_{t+j}) = E(h_t h_{t-j}) \equiv \Gamma_{hh}(-j)$. Hence $Var^*(\tilde{S}^*_{2P.1}) \xrightarrow{p} \Omega_{2.1}$.

For $Var^*(\tilde{S}^*_{2P.2})$, we first rewrite $\tilde{S}^*_{2P.2}$ as

$$\tilde{S}^*_{2P.2} = P^{-1/2} \sum_{i=1}^{P-1} a_{R,i} (h^*_{R+i} - E^* h^*_{R+i}) = P^{-1/2} \sum_{t=R+1}^{T+\tau} c_t (h^*_t - E^* h^*_t).$$

where

$$c_{R+i} = \begin{cases} a_{R,i} \text{ for } 1 \le i \le P-1 \\ 0 \text{ if } P \le i \le P-1+\tau. \end{cases}$$

By exploiting the independence between blocks, we write

$$Var^* \left(P^{-1/2} \sum_{t=R+1}^{T+\tau} c_t (h^*_t - E^* h^*_t) \right) = \underbrace{P^{-1} \sum_{i=1}^{k_2} Var^* \left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1} c_t (h^*_t - E^* h^*_t) \right)}_{\mathcal{V}}$$

Now, we show $\mathcal{V} \xrightarrow{p} \Omega_{2.2}$. Adding and subtracting appropriately, we write $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3$ where

$$\mathcal{V}_1 = P^{-1} \sum_{i=1}^{k_2} \left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1} c^2_t \Gamma_{hh}(0) + 2 \sum_{j=1}^{l-1} \sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1-j} c_t c_{t+j} \Gamma_{hh}(j) \right)$$

$$\mathcal{V}_2 = P^{-1} \sum_{i=1}^{k_2} \sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1} c^2_t \Big(Var^*(h^*_t) - \Gamma_{hh}(0) \Big)$$

$$\mathcal{V}_3 = 2 \sum_{j=1}^{l-1} P^{-1} \sum_{i=1}^{k_2} \sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1-j} c_t c_{t+j} \Big(Cov^*(h^*_t, h^*_{t+j}) - \Gamma_{hh}(j) \Big)$$

where $\mathcal{V}_2$ and $\mathcal{V}_3$ goes to zero in probability. We show $\mathcal{V}_2 = o_p(1)$; similar arguments apply to proving $\mathcal{V}_3 = o_p(1)$. For $i = 1, \ldots, k_2$, we let $m_i = R + 1 + (i-1)l$. Using this notation, we can bound $\mathcal{V}_2$ as follows

$$\mathcal{V}_2 = P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} c^2_{m_i+(j-1)} \Big(Var^*(h^*_{m_i+(j-1)}) - \Gamma_{hh}(0) \Big) \le P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} c^2_{m_i+(j-1)} \left| Var^*(h^*_{m_i+(j-1)}) - \Gamma_{hh}(0) \right|.$$

A sufficient condition for $\mathcal{V}_2 = o_p(1)$ is $\left(P^{-1} \sum_{t=R+1}^{T+\tau} c^2_t \right) E \left| Var^*(h^*_{m_i+(j-1)}) - \Gamma_{hh}(0) \right| \to 0$ where

$$P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} c^2_{m_i+(j-1)} = P^{-1} \sum_{t=R+1}^{T+\tau} c^2_t \to 2[1 - \pi^{-1} \ln(1+\pi)] - \pi^{-1} \ln(1+\pi)$$

by equation (A1-1b) in Lemma A.5 of West (1996). Thus, we only need to show $E\left|Var^*(h^*_{m_i+(j-1)})-\Gamma_{hh}(0)\right|\to 0$. Using Jensen's inequality,

$$\begin{aligned}E\left|Var^*(h^*_{m_i+(j-1)})-\Gamma_{hh}(0)\right| &\leq \left[E\left(Var^*(h^*_{m_i+(j-1)})-\Gamma_{hh}(0)\right)^2\right]^{1/2}\\ &\leq \left[Var\left(Var^*(h^*_{m_i+(j-1)})\right)+\left(EVar^*(h^*_{m_i+(j-1)})-\Gamma_{hh}(0)\right)^2\right]^{1/2}\end{aligned}$$

where for $i=1,\ldots,k_2$,

$$Var^*(h_{m_i+(j-1)})=\frac{1}{P+\tau-l}\sum_{t=R+1}^{T+\tau-l+1}\left(h_{t+(j-1)}-\mathcal{C}_{h,j}\right)^2 \text{ with } \mathcal{C}_{h,j}=\frac{1}{P+\tau-l}\sum_{s=R+1}^{T+\tau-l+1}h_{s+(j-1)}.$$

Using the uniform fourth moment bound on h_t, $Var\left(Var^*(h^*_{m_i+(j-1)})\right)\to 0$. Note that

$$EVar^*(h^*_{m_i+(j-1)})-\Gamma_{hh}(0)=-E(\mathcal{C}_{h,j})^2\leq O\left(\frac{1}{P+\tau-l}\right)\to 0$$

which completes the proof of $\mathcal{V}_2=o_p(1)$. Similar proofs can be found in the proof of Lemma A3 of Corradi and Swanson (2003) equation (38)-(39) or equation A.9 of Corradi and Swanson (2007).

Since $\mathcal{V}_2$ and $\mathcal{V}_3$ both converge to zero in probability, we only need to focus on $\mathcal{V}_1$. Adding and subtracting, we can write $\mathcal{V}_1=\mathcal{V}_{1.1}+\mathcal{V}_{1.2}$ where

$$\begin{aligned}\mathcal{V}_{1.1}=&P^{-1}\sum_{i=1}^{k_2}\left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1}c_t^2\Gamma_{hh}(0)+2\sum_{j=1}^{l-1}\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1}c_t^2\Gamma_{hh}(j)\right)\\ \mathcal{V}_{1.2}=&P^{-1}\sum_{i=1}^{k_2}\left(2\sum_{j=1}^{l-1}\left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1-j}c_tc_{t+j}-\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1}c_t^2\right)\Gamma_{hh}(j)\right),\end{aligned}$$

where $\mathcal{V}_{1.2}$ converges to zero. This step can be shown by using an argument similar to the proof of West (1996), equation (A-1b). In particular, for $i=1,...,k_2$, we can write

$$2l^{-1}\sum_{j=1}^{l-1}\left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1-j}c_tc_{t+j}-\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1}c_t^2\right)\Gamma_{hh}(j)\to 0.$$

Now, we are only left with $\mathcal{V}_{1.1}$ and

$$\mathcal{V}_{1.1}=\left(P^{-1}\sum_{t=R+1}^{T+\tau}c_t^2\right)\left(\sum_{-l+1}^{l-1}\Gamma_{hh}(j)\right),$$

where

$$P^{-1}\sum_{t=R+1}^{T+\tau}c_t^2=P^{-1}\sum_{i=1}^{P-1}a_{R,i}^2\to 2[1-\pi^{-1}\ln(1+\pi)]-\pi^{-1}\ln(1+\pi).$$

Hence,

$$\lim_{R,P\to\infty}Var^*\left(S^*_{2P.2}\right)=\lim_{R,P\to\infty}\mathcal{V}+o_p(1)\xrightarrow{p}\lim_{R,P\to\infty}\mathcal{V}_{1.1}=\Omega_{2.2}.$$

Part (c). Using the results of part (a) and (b) of this lemma, we can write

$$\lim_{R,P\to\infty} Cov^*(S_{1P}, S_{2P}) = \lim_{R,P\to\infty} Cov^*\Big(\tilde{S}_{1P}, (\tilde{S}^*_{2P.1} + \tilde{S}^*_{2P.2})\Big).$$

Exploiting the independence between $\{\gamma_{1+\tau}, \ldots, \gamma_R\}$ and $\{\eta_{R+1}, \ldots, \eta_{T+\tau}\}$, we can write

$$Cov^*\Big(\tilde{S}_{1P}, (\tilde{S}^*_{2P.1} + \tilde{S}^*_{2P.2})\Big) = Cov^*\left(\tilde{S}^*_{1P}, \tilde{S}^*_{2P.2}\right)$$

where $Cov^*\left(\tilde{S}^*_{1P}, \tilde{S}^*_{2P.1}\right) = 0$. Using the notation c_{R+i}, see definition in Part (b), we can write

$$Cov^*(\tilde{S}^*_{1P}, \tilde{S}^*_{2P.2}) = P^{-1} Cov^*\left(\sum_{t=R+1}^{T+\tau} (f^*_{t|r'} - E^* f^*_{t|r'}), \sum_{t=R+1}^{T+\tau} c_t (h^*_t - E^* h^*_t)\right).$$

Exploiting the independence between blocks, we write

$$\begin{aligned}
&P^{-1} Cov^*\left(\sum_{t=R+1}^{T+\tau} (f^*_{t|r'} - E^* f^*_{t|r'}), \sum_{t=R+1}^{T+\tau} c_t (h^*_t - E^* h^*_t)\right) \\
&= \underbrace{P^{-1} \sum_{i=1}^{k_2} Cov^*\left(\sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1} (f^*_{t|r'} - E^* f^*_{t|r'}), \sum_{t=R+1+(i-1)l}^{R+1+(i-1)l+l-1} c_t (h^*_t - E^* h^*_t)\right)}_{\mathcal{W}}
\end{aligned}$$

For notation simplicity, we let $b_i = R + 1 + (i-1)l$ for $i = 1, \ldots, k_2$ then $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$ where

$$\begin{aligned}
\mathcal{W}_1 &= P^{-1} \sum_{i=1}^{k_2} \sum_{m=1}^{l-1} \sum_{j=1}^{l} Cov^*\left(f^*_{b_i+(j-1)|r'},\ c_{b_i+(j-1)+m} h^*_{b_i+(j-1)+m}\right), \\
\mathcal{W}_2 &= P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} Cov^*\left(f^*_{b_i+(j-1)|r'},\ c_{b_i+(j-1)} h^*_{b_i+(j-1)}\right), \\
\mathcal{W}_3 &= P^{-1} \sum_{i=1}^{k_2} \sum_{m=1}^{l-1} \sum_{j=1}^{l} Cov^*\left(f^*_{b_i+(j-1)+m|r'},\ c_{b_i+(j-1)} h^*_{b_i+(j-1)}\right).
\end{aligned}$$

Adding and subtracting appropriately, we have

$$\begin{aligned}
\mathcal{W}_1 &= \mathcal{W}_{1.1} + (\mathcal{W}_1 - \mathcal{W}_{1.1}) \text{ where } \mathcal{W}_{1.1} = P^{-1} \sum_{i=1}^{k_2} \sum_{m=1}^{l-1} \sum_{j=1}^{l} c_{b_i+(j-1)+m} \Gamma_{fh}(-m), \\
\mathcal{W}_2 &= \mathcal{W}_{2.1} + (\mathcal{W}_2 - \mathcal{W}_{2.1}) \text{ where } \mathcal{W}_{2.1} = P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} c_{b_i+(j-1)} \Gamma_{fh}(0) \\
\mathcal{W}_3 &= \mathcal{W}_{3.1} + (\mathcal{W}_3 - \mathcal{W}_{3.1}) \text{ where } \mathcal{W}_{3.1} = P^{-1} \sum_{i=1}^{k_2} \sum_{m=1}^{l-1} \sum_{j=1}^{l} c_{b_i+(j-1)} \Gamma_{fh}(m)
\end{aligned}$$

where $\Gamma_{fh}(m) = E(f_{t|r'} h_{t+m}) = E(f_{t|r'} h_{t-m}) = \Gamma_{fh}(-m)$. Note that $|\mathcal{W}_1 - \mathcal{W}_{1.1}| = o_p(1)$, $|\mathcal{W}_2 - \mathcal{W}_{2.1}| = o_p(1)$ and $|\mathcal{W}_3 - \mathcal{W}_{3.1}| = o_p(1)$ by using the same arguments that proves the results of part (b). Further adding and subtracting on $\mathcal{W}_{1.1}$ and $\mathcal{W}_{3.1}$, we can write

$$\mathcal{W}_{1.1} = \mathcal{W}_{1.1.1} + (\mathcal{W}_{1.1} - \mathcal{W}_{1.1.1}) \text{ where } \mathcal{W}_{1.1.1} = \sum_{m=1}^{l-1} P^{-1} \sum_{i=1}^{k_2} \sum_{j}^{l} c_{b_i+(j-1)} \Gamma_{fh}(-m),$$

$$\mathcal{W}_{3.1} = \mathcal{W}_{3.1.1} + (\mathcal{W}_{3.1} - \mathcal{W}_{3.1.1}) \text{ where } \mathcal{W}_{3.1.1} = \sum_{m=1}^{l-1} P^{-1} \sum_{i=1}^{k_2} \sum_{j=1}^{l} c_{b_i+(j-1)} \Gamma_{fh}(m)$$

Note that $|\mathcal{W}_{1.1} - \mathcal{W}_{1.1.1}| = o(1)$ and $|\mathcal{W}_{3.1} - \mathcal{W}_{3.1.1}| = o(1)$ by an argument similar to that used in the proof of West's (1996) Lemma A.6. Hence,

$$Cov^* \left(\tilde{S}_{1P}^*, \tilde{S}_{2P.2}^* \right) \xrightarrow{p} P^{-1} \left(\sum_{i=1}^{P-1} a_{R,i} \right) \sum_{m=-l+1}^{l-1} \Gamma_{fh}(m) \to \Omega_{12}$$

where $P^{-1} \left(\sum_{i=1}^{P-1} a_{R,i} \right) \to 1 - \pi^{-1} \ln(1+\pi)$, see remark in part (b) or by Lemma A6 of West (1996).
■

A.2 Proofs of results in the paper

Proof of Lemma 4.1. Given Lemma A.1, by two mean value expansions of $f_{t+\tau|r'}(\hat{\beta}_t)$ and $f_{t+\tau|r'}(\hat{\beta}(t))$ around β_0, we can write

$$\hat{S}_P - \tilde{S}_P = FP^{-1/2} \sum_{t=R}^{T} (\hat{\beta}(t) - \hat{\beta}_t) + o_p(1).$$

The result follows by showing that $P^{-1/2} \sum_{t=R}^{T} (\hat{\beta}(t) - \hat{\beta}_t) = o_p(1)$. Using the definitions of $\hat{\beta}_t$ and $\hat{\beta}(t)$, we can write

$$P^{-1/2} \sum_{t=R}^{T} (\hat{\beta}(t) - \hat{\beta}_t) = \sum_{i=1}^{3} \mathcal{C}_i,$$

where $\mathcal{C}_1 = P^{-1/2} \sum_{t=R}^{T} \left(\hat{B}(t) - B(t) \right) H(t)$, $\mathcal{C}_2 = P^{-1/2} \sum_{t=R}^{T} B(t) \left(\hat{H}(t) - H(t) \right)$, and $\mathcal{C}_3 = P^{-1/2} \sum_{t=R}^{T} \left(\hat{B}(t) - B(t) \right) \left(\hat{H}(t) - H(t) \right)$. Next, we show $\mathcal{C}_i = o_p(1)$ for $i = 1, 2, 3$. Starting with $\mathcal{C}_1$,

$$\mathcal{C}_1 \leq \sup_t \left| \hat{B}(t) - B(t) \right| P^{-1/2} \sum_{t=R}^{T} |H(t)|,$$

where $\sup_t \left| \hat{B}(t) - B(t) \right| = o_p(1)$ by Lemma A.1 (a) and $P^{-1/2} \sum_{t=R}^{T} |H(t)| = O_p(1)$ by the proof of West's (1996) Lemma A.4 (c). Next, adding and subtracting appropriately,

$$\mathcal{C}_2 = P^{-1/2} \sum_{t=R}^{T} B \left(\hat{H}(t) - H(t) \right) + P^{-1/2} \sum_{t=R}^{T} (B(t) - B) \left(\hat{H}(t) - H(t) \right).$$

It follows that

$$\mathcal{C}_2 \leq BP^{-1/2} \sum_{t=R}^{T} \left(\hat{H}(t) - H(t) \right) + \sup_t |B(t) - B| \, P^{-1/2} \sum_{t=R}^{T} |\hat{H}(t) - H(t)|$$

where $P^{-1/2}\sum_{t=R}^{T}|\hat{H}(t)-H(t)|=o_p(1)$ by Lemma A.1 (b), and $\sup_t|B(t)-B|=o_p(1)$ by Assumption 2 (a). We can show that $\mathcal{C}_3=o_p(1)$ by a similar argument, completing the proof. ■

Proof of Lemma 4.2. Part (a) follows from eq. (A-1.c) in Lemma A5 in West (1996), whereas part (b) follows from West's (1996) Lemma A2 (a). ■

Proof of Lemma 5.1. This result is obtained by taking the difference of two second-order mean value expansions. The first expansion expands $f^*_{t+\tau|r'}(\hat{\beta}^*_t)$ around β_0, whereas the second expansion expands $f_{t+\tau|r'}(\bar{\beta}_t)$ around β_0, where $\bar{\beta}_t\equiv t^{-1}R\hat{\beta}_R+t^{-1}(t-R)\hat{\beta}_P$, with $\hat{\beta}_R\equiv(R^{-1}\sum_{s=1+\tau}^{R}x_{s-\tau}x'_{s-\tau})^{-1}R^{-1}\sum_{s=1+\tau}^{R}x_{s-\tau}y_s$ and $\hat{\beta}_P\equiv(P^{-1}\sum_{s=R+\tau}^{T+\tau}x_{s-\tau}x'_{s-\tau})^{-1}P^{-1}\sum_{s=R+\tau}^{T+\tau}x_{s-\tau}y_s$.

More specifically, we have that

$$P^{-1/2}\sum_{t=R}^{T}f^*_{t+\tau|r'}\left(\hat{\beta}^*_t\right)=P^{-1/2}\sum_{t=R}^{T}f^*_{t+\tau|r'}+\xi^*_1+\xi^*_2$$

where

$$\xi^*_1\equiv P^{-1/2}\sum_{t=R}^{T}f^*_{t+\tau|r',\beta}(\hat{\beta}^*_t-\beta_0)\ \text{ and }\ \xi^*_2\equiv 0.5P^{-1/2}\sum_{t=R}^{T}\frac{\partial^2}{\partial\beta^2}f^*_{t+\tau|r'}(\tilde{\beta}^*_t)(\hat{\beta}^*_t-\beta_0)^2,$$

where $\tilde{\beta}^*_t$ lies between $\hat{\beta}^*_t$ and β_0, and we recall that $f^*_{t+\tau|r'}\equiv f^*_{t+\tau|r'}(\beta_0)$ and $f^*_{t+\tau|r',\beta}\equiv f^*_{t+\tau|r',\beta}(\beta_0)$. To show $\xi^*_2=o^*_p(1)$, note that

$$|\xi^*_2|\leq 0.5\left(\sup_t\left|P^{1/4}\left(\hat{\beta}^*_t-\beta_0\right)\right|\right)^2P^{-1}\sum_{t=R}^{T}\left|\frac{\partial^2}{\partial\beta^2}f^*_{t+\tau|r'}\left(\tilde{\beta}^*_t\right)\right|.$$

The result follows by Lemma A.2 (c) and the fact that we can show that $P^{-1}\sum_{t=R}^{T}|\frac{\partial^2}{\partial\beta^2}f^*_{t+\tau|r'}\left(\tilde{\beta}^*_t\right)|=O^*_p(1)$, as we argue next. By Assumption 1, and the fact that $\tilde{\beta}^*_t\xrightarrow{P^*}\beta_0$, we can bound $|\frac{\partial^2}{\partial\beta^2}f^*_{t+\tau|r'}\left(\tilde{\beta}^*_t\right)|$ by $\sup_{\beta\in N}|\frac{\partial^2}{\partial\beta^2}f_{\eta_{t+\tau}|r'}(\beta)|\leq m_{\eta_{t+\tau}}\equiv m^*_{t+\tau}$. The result follows by Markov's inequality,

$$P^*\left(P^{-1}\sum_{t=R}^{T}\left|\frac{\partial^2}{\partial\beta^2}f^*_{t+\tau|r'}(\tilde{\beta}^*_t)\right|>\delta\right)\leq P^*\left(P^{-1}\sum_{t=R}^{T}m^*_{t+\tau}>\delta\right)\leq\delta^{-1}P^{-1}\sum_{t=R}^{T}E^*(m^*_{t+\tau}),$$

since $P^{-1}\sum_{t=R}^{T}E^*(m^*_{t+\tau})=O_p(1)$ by the properties of the MBB expectation. For ξ^*_1, adding and subtracting appropriately yields

$$\xi^*_1=P^{-1/2}\sum_{t=R}^{T}f^*_{t+\tau|r',\beta}\left(\hat{\beta}^*_t-\beta_0\right)=\sum_{i=1}^{4}\xi^*_{1.i},$$

where

$$\xi^*_{1.1}=FBP^{-1/2}\sum_{t=R}^{T}H^*(t),\quad \xi^*_{1.2}=P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta}-F)BH^*(t)$$

$$\xi^*_{1.3}=P^{-1/2}\sum_{t=R}^{T}F(B^*(t)-B)H^*(t),\text{ and }\xi^*_{1.4}=P^{-1/2}\sum_{t=R}^{T}(f^*_{t+\tau|r',\beta}-F)(B^*(t)-B)H^*(t).$$

By Lemma A.2 (d),(e) and (f), $\xi^*_{1,i} = o^*_p(1)$ for $i = 2, 3, 4$, respectively. Hence,

$$P^{-1/2}\sum_{t=R}^{T} f^*_{t+\tau|r'}(\hat{\beta}^*_t) = P^{-1/2}\sum_{t=R}^{T} f^*_{t+\tau|r'} + FBP^{-1/2}\sum_{t=R}^{T} H^*(t) + o^*_p(1). \tag{5}$$

Similarly, an expansion of $f_{t+\tau|r'}(\bar{\beta}_t)$ around β_0 yields

$$P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r'}(\bar{\beta}_t) = P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r'} + \bar{\xi}_1 + \bar{\xi}_2,$$

where

$$\bar{\xi}_1 = P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r',\beta}(\bar{\beta}_t - \beta_0) \text{ and } \bar{\xi}_2 = 0.5P^{-1/2}\sum_{t=R}^{T} \frac{\partial^2}{\partial\beta^2} f_{t+\tau|r'}(\ddot{\beta}_t)(\bar{\beta}_t - \beta_0)^2.$$

where $\ddot{\beta}_t$ lies between $\bar{\beta}_t$ and β_0, and $f_{t+\tau|r',\beta} \equiv f_{t+\tau|r',\beta}(\beta_0)$. We can show that $\bar{\xi}_2 = o_p(1)$ using a similar argument to that used to show that $\xi^*_2 = o^*_p(1)$. In particular, it suffices to show that $\sup_t \left|P^{1/4}\left(\bar{\beta}_t - \beta_0\right)\right| = o_p(1)$ and $P^{-1}\sum_{t=R}^{T}\left|\frac{\partial^2}{\partial\beta^2} f_{t+\tau|r'}\left(\ddot{\beta}_t\right)\right| = O_p(1)$. For $\bar{\xi}_1$, note that by definition we can write $\bar{\beta}_t - \beta_0 = \frac{R}{t}(\hat{\beta}_R - \beta_0) + \frac{t-R}{t}(\hat{\beta}_P - \beta_0)$, where

$$\hat{\beta}_R - \beta_0 = B(R)\, H(R),$$

with $B(R) \equiv \left(R^{-1}\sum_{s=1+\tau}^{R} x_s x_s'\right)^{-1}$, $H(R) = R^{-1}\sum_{s=1+\tau}^{R} h_s$, using our previous definitions of $B(t)$ and $H(t)$. Similarly, given the definition of $\hat{\beta}_P$, we can write

$$\hat{\beta}_P - \beta_0 = \underbrace{\left(P^{-1}\sum_{s=R+\tau}^{T+\tau} x_s x_s'\right)^{-1}}_{\equiv B_P} \underbrace{P^{-1}\sum_{s=R+\tau}^{T+\tau} h_s}_{\equiv H_P = \bar{h}_P}.$$

With this notation, we have that

$$\bar{\xi}_1 = P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r',\beta}\left(\frac{R}{t}B(R)\, H(R) + \frac{t-R}{t}B_P H_P\right).$$

Adding and subtracting appropriately, we rewrite $\bar{\xi}_1$ as $\bar{\xi}_1 = \sum_{i=1}^{8}\bar{\xi}_{1.i}$ where

$$\bar{\xi}_{1.1} = FBP^{-1/2}\sum_{t=R}^{T}\frac{R}{t}H(R)\ ,\ \bar{\xi}_{1.2} = FBP^{-1/2}\sum_{t=R}^{T}\frac{t-R}{t}H_P$$

$$\bar{\xi}_{1.3} = FP^{-1/2}\sum_{t=R}^{T}\frac{R}{t}\big(B(R) - B\big)H(R)\ ,\ \bar{\xi}_{1.4} = FP^{-1/2}\sum_{t=R}^{T}\frac{t-R}{t}\big(B_P - B\big)H_P$$

$$\bar{\xi}_{1.5} = P^{-1/2}\sum_{t=R}^{T}\frac{R}{t}\big(f_{t+\tau|r',\beta} - F\big)BH(R)\ ,\ \bar{\xi}_{1.6} = P^{-1/2}\sum_{t=R}^{T}\frac{t-R}{t}\big(f_{t+\tau|r',\beta} - F\big)BH_P$$

$$\bar{\xi}_{1.7} = P^{-1/2}\sum_{t=R}^{T}\frac{R}{t}\big(f_{t+\tau|r',\beta} - F\big)\big(B(R) - B\big)H(R)\ ,\ \bar{\xi}_{1.8} = P^{-1/2}\sum_{t=R}^{T}\frac{t-R}{t}\big(f_{t+\tau|r',\beta} - F\big)\big(B_P - B\big)H_P.$$

We can show that $\bar{\xi}_{1.3},\ \bar{\xi}_{1.5}$ and $\bar{\xi}_{1.7}$ are $o_p(1)$ by applying arguments similar to those in West (1996) (cf. his Lemma A.4). Similar proofs show that $\bar{\xi}_{1.4},\ \bar{\xi}_{1.6}$ and $\bar{\xi}_{1.8}$ are also $o_p(1)$. Hence, we obtain that $\bar{\xi}_1 = \bar{\xi}_{1.1} + \bar{\xi}_{1.2} + o_p(1)$ and

$$P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r'}(\bar{\beta}_t) = P^{-1/2}\sum_{t=R}^{T} f_{t+\tau|r'} + \bar{\xi}_{1.1} + \bar{\xi}_{1.2} + o_p(1). \tag{6}$$

Subtracting (5) from (6) yields

$$\tilde{S}_P^* \equiv P^{-1/2}\sum_{t=R}^{T}\left(f_{t+\tau|r'}^*(\hat{\beta}_t^*) - f_{t+\tau|r'}(\bar{\beta}_t)\right) = P^{-1/2}\sum_{t=R}^{T}\left(f_{t+\tau|r'}^* - f_{t+\tau|r'}\right) + \left(\xi_{1.1}^* - \bar{\xi}_{1.1} - \bar{\xi}_{1.2}\right) + o_p(1),$$

where

$$P^{-1/2}\sum_{t=R}^{T}\left(f_{t+\tau|r'}^* - f_{t+\tau|r'}\right) = S_{1P}^*,$$

and $\xi_{1.1}^* - \bar{\xi}_{1.1} - \bar{\xi}_{1.2} = FBS_{2P}^*$, as we show next. In particular, using arguments similar to those of Lemma A.5 of West (1996), we have that

$$\xi_{1.1}^* \equiv FBP^{-1/2}\sum_{t=R}^{T} H^*(t) = FBa_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R} h_s^* + FBP^{-1/2}\sum_{i=1}^{P-1} a_{R,i}h_{R+i}^*,$$

where $a_{R,i} \equiv (R+i)^{-1} + \ldots + (R+P-1)^{-1}$ for $0 \le i \le P-1$. Moreover, we can rewrite $\bar{\xi}_{1.1}$ as follows,

$$\bar{\xi}_{1.1} = FBP^{-1/2}\underbrace{\left(\sum_{t=R}^{T} t^{-1}\right)}_{\equiv a_{R,0}} R\underbrace{H(R)}_{\equiv \bar{h}_R} = \frac{R}{R-\tau}FBP^{-1/2}a_{R,0}(R-\tau)\,\bar{h}_R = FBa_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}\bar{h}_R + o_p(1),$$

where the second equality holds by $H(R) \equiv \bar{h}_R$ and the fact that $a_{R,0} = \sum_{t=R}^{T} t^{-1}$, and the third equality follows from $R/(R-\tau) \to 1$. Lastly, we can rewrite

$$\bar{\xi}_{1.2} = FBP^{-1/2}\left(\sum_{t=R}^{T}\frac{t-R}{t}\right)H_P = FBP^{-1/2}\underbrace{\left(\frac{1}{R+1} + \ldots + \frac{P-1}{T}\right)}_{=\sum_{i=1}^{P-1} a_{R,i}}\bar{h}_P = FBP^{-1/2}\left(\sum_{i=1}^{P-1} a_{R,i}\right)\bar{h}_P.$$

Hence,

$$\xi_{1.1}^* - \bar{\xi}_{1.1} - \bar{\xi}_{1.2} = FBa_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h_s^* - \bar{h}_R) + FBP^{-1/2}\sum_{i=1}^{P-1} a_{R,i}(h_{R+i}^* - \bar{h}_P) \equiv FBS_{2P}^*.$$

■

Theorem 5.1. Theorem 5.1 follows from Polya's theorem (e.g., Serfling, 1980 Chapter 1.5.3 page 18) if $\Omega^{-1/2}\tilde{S}_P^{\mu} \xrightarrow{d} N(0,1)$ and $\Omega^{-1/2}\tilde{S}_P^* \xrightarrow{d^*} N(0,1)$. Using the expansion in Lemma 4.1, we get $\tilde{S}_P^{\mu} \xrightarrow{d} N(0,\Omega)$ by Theorem 4.1 of West (1996). We are left to show that $\tilde{S}_P^* \xrightarrow{d^*} N(0,\Omega)$. Recall that $\tilde{S}_{1P}^* = P^{-1/2}\sum_{t=R+1}^{T+\tau}(f_{t|r'}^* - E^* f_{t|r'}^*)$, $\tilde{S}_{2P.1}^* = a_{R,0}P^{-1/2}\sum_{s=1+\tau}^{R}(h_s^* - E^* h_s^*)$, and $\tilde{S}_{2P.2}^* = P^{-1/2}\sum_{t=R+1}^{T+\tau} c_t(h_t^* - E^* h_t^*)$ with

$$c_{R+i} = \begin{cases} a_{R,i} \text{ for } 1 \leq i \leq P-1 \\ 0 \text{ if } P \leq i \leq P-1+\tau. \end{cases}$$

Using Lemma 5.1 and the results in Lemma A.4, we can write,

$$\begin{aligned} \tilde{S}_P^* &= \tilde{S}_{1P}^* + FB\left(\tilde{S}_{2P.1}^* + \tilde{S}_{2P.2}^*\right) + o_p^*(1) \\ &= \underbrace{(\tilde{S}_{1P}^* + FB\tilde{S}_{2P.2}^*)}_{\equiv \mathcal{K}_1^*} + \underbrace{FB\tilde{S}_{2P.1}^*}_{\equiv \mathcal{K}_2^*} + o_p^*(1). \end{aligned}$$

This alternative representation of $\tilde{S}_P^*$ enable us to separate the whole bootstrap statistic into two independent, zero bootstrap mean, bootstrap statistics $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$. The independence of these two terms can be seen from bootstrap algorithm in section 5. Recall that $\tilde{S}_{2P.1}^*$ is based on random indexes $I_1, \ldots, I_{k_1}$ whereas $\tilde{S}_{1P}^*$ and $\tilde{S}_{2P.2}^*$ are both based on random indexes $J_1, \ldots, J_{k_2}$, which are independent from $I_1, \ldots, I_{k_1}$. Next we show $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ both converge to normal distribution with zero mean and appropriate limiting variances. Without loss of generality, we let $\tau = 1$ for the rest of the proof.

For $\mathcal{K}_2^*$, we write

$$\mathcal{K}_2^* = FBa_{R,0}\left(\frac{P}{R-1}\right)^{-1/2}(R-1)^{-1/2}\sum_{s=2}^{R}(h_s^* - E^* h_s^*)$$

where $\left(Var\left((R-1)^{-1/2}\sum_{s=2}^{R} h_s\right)\right)^{-1/2}(R-1)^{-1/2}\sum_{s=2}^{R}(h_s^* - E^* h_s^*) \xrightarrow{d^*} N(0,1)$ by Corollary 3.1 of Fitzenberger (1998). Using the result of $Var^*(\tilde{S}_{2P.1}^*)$ in Lemma A.4 (b), we can write

$$Var^*\left(\mathcal{K}_2^*\right) \xrightarrow{p} F^2B^2\Omega_{2.1}.$$

where the definition of $\Omega_{2.1}$ can be found in Lemma 4.2. Hence, $\mathcal{K}_2^*$ converge to normal distribution with zero mean and variance $F^2B^2\Omega_{2.1}$.

For $\mathcal{K}_1^*$, we first simplify the notation by letting $m_i = R+1+(i-1)l$ then write $\mathcal{K}_1^*$ as sum of independent block sums, i.e., $\mathcal{K}_1^* = P^{1/2}k_2^{-1}\sum_{i=1}^{k_2}\tilde{U}_{m_i}^*$ where

$$\tilde{U}_{m_i}^* = l^{-1}\sum_{j=1}^{l}\left(\tilde{f}_{m_i+(j-1)|r'}^* + FBc_{m_i+(j-1)}\tilde{h}_{m_i+(j-1)}^*\right)$$

with $\tilde{f}_{m_i+(j-1)|r'}^* = f_{m_i+(j-1)|r'}^* - E^* f_{m_i+(j-1)|r'}^*$ and $\tilde{h}_{m_i+(j-1)}^* = h_{m_i+(j-1)}^* - E^* h_{m_i+(j-1)}^*$. Using results in Lemma A.4, we can write

$$Var^*(\mathcal{K}_1^*) \xrightarrow{p} \underbrace{\Omega_1 + F^2B^2\Omega_{2,2} + 2FB\Omega_{12}}_{\equiv \Sigma}$$

where the definition of Ω_1, $\Omega_{2,2}$, and Ω_{12} can be found in equation (1) and Lemma 4.2 respectively. Then $\Sigma^{-1/2}\mathcal{K}_1^* = \sum_{i=1}^{k} \tilde{Z}_{m_i}$ where $\tilde{Z}_{m_i} = \Sigma^{-1/2}P^{-1/2}l\tilde{U}_{m_i}^*$. Noting that $\{\tilde{Z}_{m_1}^*, \ldots, \tilde{Z}_{m_{k_2}}^*\}$ is a zero mean, independent heterogeneous sequence which satisfies a CLT for independent heterogeneous sequence (see, 23.6 Lindeberg's Theorem and 23.11 Liapunov's Theorem in Davidson 1994). We verify Liapunov's condition using arguments similar to those in Goncalves and White (2002) (see equation (A.5) on p. 1384). ∎

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